SUPPLEMENTS FOR
MA 131

1. DIFFERENCE EQUATIONS AND THE MATHEMATICS OF FINANCE

Chapter 10 of the text **Finite Mathematics and its Applications** by L.J Goldstein and D.I. Schneider.

2. COMPUTATIONAL MATHEMATICS

Chapters 3, 4, and 14 of **Computational Mathematics for MA 132** by R.E. White, Department of Mathematics, NCSU, white@math.ncsu.edu.

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Chapter 10

DIFFERENCE EQUATIONS AND THE MATHEMATICS OF FINANCE

In this chapter we discuss a number of topics from the mathematics of finance—compound interest, mortgages, and annuities. As we shall see, all such financial transactions can be described by a single type of equation, called a *difference equation*. Furthermore, the same type of difference equation can be used to model many other phenomena, such as the spread of information, radioactive decay, and population growth, to mention a few.

10.1. Introduction to Difference Equations, I

In order to understand what difference equations are and how they arise, consider two examples concerned with financial transactions in a savings account.

EXAMPLE 1 Suppose that a savings account initially contains \$40 and earns 6% interest, compounded annually. Determine a formula which describes how to compute each year's balance from the previous year's balance.

Solution The balances in the account for the first few years can be given as in the chart below, where y_0 is the initial balance (balance after zero years), y_1 is the balance after one year, and so on.

Year	*Balance*	*Interest for year*
0	$y_0 = \$40$	$(.06)40 = 2.40$
1	$y_1 = 42.40$	$(.06)42.40 = 2.54$
2	$y_2 = 44.94$	$(.06)44.94 = 2.70$
3	$y_3 = 47.64$	

Once the balance is known for a particular year, the balance at the end of the next year is computed as follows:

$$[\text{balance at end of next year}] = [\text{balance at end of this year}] + [\text{interest on balance at end of this year}].$$

That is,

$$y_1 = y_0 + .06y_0$$
$$y_2 = y_1 + .06y_1$$
$$y_3 = y_2 + .06y_2.$$

(Notice that since $3 = 2 + 1$, the last equation is $y_{2+1} = y_2 + .06y_2$.) In general, if y_n is the balance after n years, then y_{n+1} is the balance at the end of the next year, and so

$$y_{n+1} = y_n + .06y_n \qquad \text{for } n = 0, 1, 2, \ldots.$$

This equation can be simplified:

$$\begin{aligned} y_{n+1} &= y_n + .06y_n \\ &= 1 \cdot y_n + .06y_n \\ &= (1 + .06)y_n \\ y_{n+1} &= 1.06y_n. \end{aligned} \tag{1}$$

In other words, the balance after $n + 1$ years is 1.06 times the balance after n years. This formula describes how the balance is computed in successive years. For instance, using this formula, we can compute y_4. Indeed, setting $n = 3$ and using the value of y_3 from the chart above,

$$y_4 = y_{3+1} = 1.06y_3 = 1.06(47.64) = \$50.50.$$

In a similar way we can compute all of the year-end balances, one after another, by using formula (1).

EXAMPLE 2 Suppose that a savings account contains \$40 and earns 6% interest, compounded annually. At the end of each year a \$3 withdrawal is made. Determine a formula which describes how to compute each year's balance from the previous year's balance.

Solution As in the preceding example, we compute the first few balances in a straightforward way and organize the data in a chart.

Balance	*+ Interest for year*	*− Withdrawal*
$y_0 = \$40$	$(.06)40 = 2.40$	\$3
$y_1 = 39.40$	$(.06)39.40 = 2.36$	\$3
$y_2 = 38.76$		

Reasoning as in the preceding example, let $y_n =$ the balance after n years. Then, by analyzing the calculations of the above chart, we see that

$$\begin{array}{ccccccc} y_{n+1} & = & y_n & + & .06y_n & - & 3 \\ \text{[new balance]} & = & \text{[old balance]} & + & \text{[interest on old balance]} & - & \text{[withdrawal]} \end{array}$$

or

$$y_{n+1} = 1.06y_n - 3. \tag{2}$$

This formula allows us to compute the values of y_n successively. For example, the chart above gives $y_2 = 38.76$. Therefore, from equation (2) with $n = 2$,

$$y_3 = (1.06)y_2 - 3 = (1.06)(38.76) - 3 = 38.09.$$

Equations of the form (1) and (2) are examples of what are called difference equations. More precisely, a difference equation is an equation of the form

$$y_{n+1} = ay_n + b,$$

where a, b are specific numbers. For example, for the difference equation (1) we have $a = 1.06$, $b = 0$. For the difference equation (2) we have $a = 1.06$, $b = -3$. A difference equation gives a procedure for calculating the term y_{n+1} from the preceding term y_n, thereby allowing one to compute all the terms—provided, of course, that a place to start is given. For this purpose one is usually given a specific value for y_0. Such a value is called an *initial value*. In both of the previous examples the initial value was 40.

Whenever we are given a difference equation, our goal is to determine as much information as possible about the terms y_0, y_1, y_2, and so on. To this end there are three things we can do:

1. *Generate the first few terms.* This is useful in giving us a feeling for how successive terms are generated.

2. *Graph the terms*. The terms which have been generated can be graphed by plotting the points $(0, y_0)$, $(1, y_1)$, $(2, y_2)$, and so on. Corresponding to the term y_n, we plot the point (n, y_n). The resulting graph (Fig. 1) depicts how the terms increase or decrease as n increases. In Figs. 2 and 3 we have drawn the graphs corresponding to the difference equations of Examples 1 and 2.

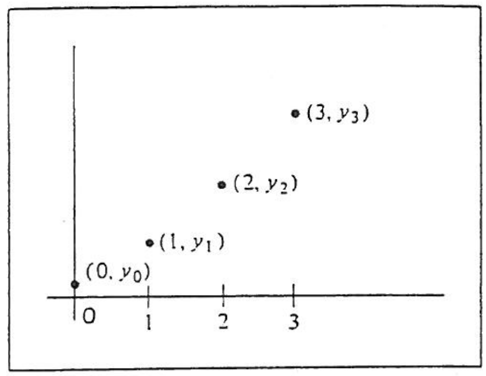


FIGURE 1

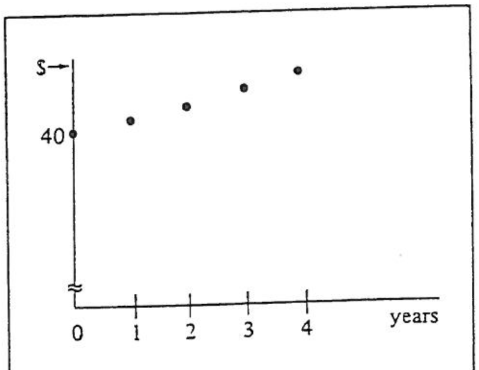


FIGURE 2 Graph of $y_{n+1} = 1.06y_n$, $y_0 = 40$.

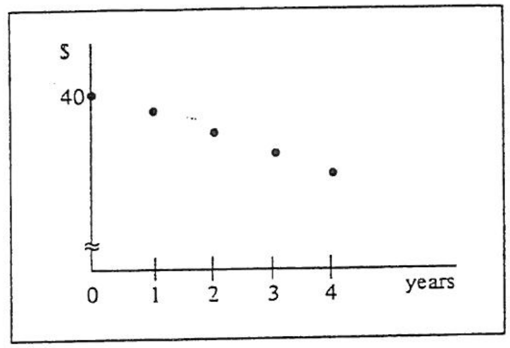


FIGURE 3 Graph of $y_{n+1} = 1.06y_n - 3$, $y_0 = 40$.

3. *Solve the difference equation*. By a *solution* of a difference equation we mean a general formula from which we can directly calculate any term without first having to calculate all of the terms preceding it. One can always write down a solution using the values of a, b, and y_0. Assume for now that $a \neq 1$. Then a solution of $y_{n+1} = ay_n + b$ is given by:

$$\boxed{y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n, \qquad a \neq 1.} \tag{3}$$

(This formula will be derived in the next section.)

EXAMPLE 3 Solve the difference equation $y_{n+1} = 1.06y_n$, $y_0 = 40$.

Solution Here $a = 1.06$, $b = 0$, $y_0 = 40$. So $b/(1 - a) = 0$, and from equation (3)

$$y_n = 0 + (40 - 0)(1.06)^n$$

$$\text{Answer:} \quad y_n = 40(1.06)^n.$$

EXAMPLE 4 Solve the difference equation $y_{n+1} = 1.06y_n - 3$, $y_0 = 40$. Determine y_{21}.

Solution Here $a = 1.06$, $b = -3$, $y_0 = 40$. So $b/(1 - a) = -3/(1 - 1.06) = -3/-.06 = 50$. Therefore,

$$y_n = 50 + (40 - 50)(1.06)^n$$

$$\text{Answer:} \quad y_n = 50 - 10(1.06)^n.$$

In particular,

$$y_{21} = 50 - 10(1.06)^{21}.$$

Using a calculator, we find that $(1.06)^{21} \approx 3.4$, so that

$$y_{21} = 50 - 10(3.4) = 50 - 34 = 16.$$

So, after 21 years, the bank account of Example 2 will contain about \$16. (If we did not have the solution $y_n = 50 - 10(1.06)^n$, we would need to perform 21 successive calculations in order to determine y_{21}.)

In the next two examples we apply our entire three-step procedure to analyze some specific difference equations.

EXAMPLE 5 Apply the three-step procedure to study the difference equation $y_{n+1} = .2y_n + 4.8$, $y_0 = 1$.

Solution First we compute a few terms:

$$\begin{aligned} y_0 &= 1 \\ y_1 &= .2(1) + 4.8 = 5 \\ y_2 &= .2(5) + 4.8 = 5.8 \\ y_3 &= .2(5.8) + 4.8 = 5.96 \\ y_4 &= .2(5.96) + 4.8 = 5.992. \end{aligned}$$

Next, we graph these terms (Fig. 4). That is, we plot (0, 1), (1, 5), (2, 5.8), (3, 5.96), (4, 5.992). Note that the values of y_n increase. Finally, we solve the difference equation. Here $a = .2$, $b = 4.8$, $y_0 = 1$. Thus,

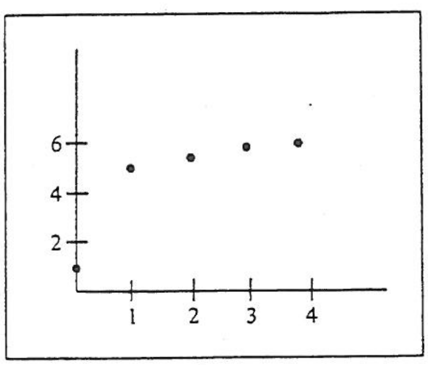


FIGURE 4 Graph of $y_{n+1} = .2y_n + 4.8$, $y_0 = 1$.

$$\frac{b}{1-a} = \frac{4.8}{1-.2} = \frac{4.8}{.8} = 6$$

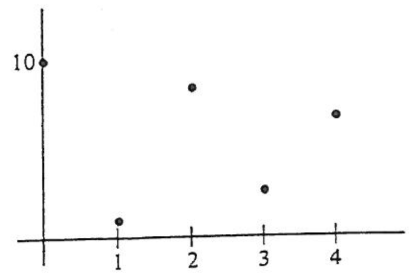
$$y_n = 6 + (1 - 6)(.2)^n = 6 - 5(.2)^n.$$

EXAMPLE 6 Apply the three-step procedure to study the difference equation $y_{n+1} = -.8y_n + 9$, $y_0 = 10$.

Solution The first few terms are

$$\begin{aligned} y_0 &= 10 \\ y_1 &= -.8(10) + 9 = -8 + 9 = 1 \\ y_2 &= -.8(1) + 9 = -.8 + 9 = 8.2 \\ y_3 &= -.8(8.2) + 9 = 2.44 \\ y_4 &= -.8(2.44) + 9 = 7.048. \end{aligned}$$

The accompanying graph corresponds to these terms. Notice that in this case the



points oscillate up and down. To solve the difference equation note that

$$\frac{b}{1-a} = \frac{9}{1-(-.8)} = \frac{9}{1.8} = 5.$$

Thus,

$$y_n = 5 + (10 - 5)(-.8)^n = 5 + 5(-.8)^n.$$

Difference equations can be used to describe many real-life situations. The solution of a particular difference equation in such instances yields a mathematical model of the situation. For example, consider the banking problem of Example 2. We described the activity in the account by a difference equation, where y_n represents the amount of money in the account after n years. Solving the difference equation, we found that

$$y_n = 50 - 10(1.06)^n.$$

And this formula gives a mathematical model of the bank account.

Here is another example of a mathematical model derived using difference equations.

EXAMPLE 7 Suppose that the population of a certain country is currently 6 million. The growth of this population attributable to an excess of births over deaths is 2% per year. Further, the country is experiencing immigration at the rate of 40,000 people per year.

(a) Find a mathematical model for the population of the country.

(b) What will be the population of the country after 35 years?

Solution (a) Let y_n denote the population (in millions) of the country after n years. Then $y_0 = 6$. The growth in the population in year $n + 1$ due to an excess of births over deaths is $.02y_n$. There are .04 (million) immigrants each year. Therefore,

$$y_{n+1} = y_n + .02y_n + .04 = 1.02y_n + .04.$$

So the terms satisfy the difference equation

$$y_{n+1} = 1.02y_n + .04, \qquad y_0 = 6.$$

Here

$$\frac{b}{1-a} = \frac{.04}{1-1.02} = \frac{.04}{-.02} = -2$$

$$y_n = -2 + (6 - (-2))(1.02)^n$$

or

$$y_n = -2 + 8(1.02)^n.$$

This last formula is our desired mathematical model for the population.

(b) To determine the population after 35 years merely compute y_{35}:

$$y_{35} = -2 + 8(1.02)^{35} \approx -2 + 8(2) = 14,$$

since $(1.02)^{35} \approx 2$. So the population after 35 years will be about 14 million.

PRACTICE PROBLEMS 1

1. Consider the difference equation $y_{n+1} = -2y_n + 21$, $y_0 = 7.5$.
   (a) Generate y_0, y_1, y_2, y_3, y_4 from the difference equation.
   (b) Graph these first few terms.
   (c) Solve the difference equation.
2. Use the solution in 1(c) to obtain the terms y_0, y_1, y_2.

EXERCISES 1

For each of the difference equations in Exercises 1–6 identify a and b and compute $b/(1 - a)$.

1. $y_{n+1} = 4y_n - 6$
2. $y_{n+1} = -3y_n + 16$
3. $y_{n+1} = -\frac{1}{2}y_n$
4. $y_{n+1} = \frac{1}{3}y_n + 4$
5. $y_{n+1} = -\frac{2}{3}y_n + 15$
6. $y_{n+1} = .5y_n - 4$

In Exercises 7–14: (a) Generate y_0, y_1, y_2, y_3, y_4 from the difference equation. (b) Graph these first few terms. (c) Solve the difference equation.

7. $y_{n+1} = \frac{1}{2}y_n - 1, \quad y_0 = 10$

8. $y_{n+1} = .5y_n + 5, \quad y_0 = 2$

9. $y_{n+1} = 2y_n - 3, \quad y_0 = 3.5$

10. $y_{n+1} = 5y_n - 32, \quad y_0 = 8$

11. $y_{n+1} = -.4y_n + 7, \quad y_0 = 17.5$

12. $y_{n+1} = -2y_n, \quad y_0 = \frac{1}{2}$

13. $y_{n+1} = 2y_n - 16, \quad y_0 = 15$

14. $y_{n+1} = 2y_n + 3, \quad y_0 = -2$

15. The solution to $y_{n+1} = .2y_n + 4.8$, $y_0 = 1$ is $y_n = 6 - 5(.2)^n$. Use the solution to compute y_0, y_1, y_2, y_3, y_4.

16. The solution to $y_{n+1} = -.8y_n + 9$, $y_0 = 10$ is $y_n = 5 + 5(-.8)^n$. Use the solution to compute y_0, y_1, y_2.

17. One thousand dollars is deposited into a savings account paying 5% interest compounded annually. Let y_n be the amount after n years. What is the difference equation showing how to compute y_{n+1} from y_n?

18. The population of a certain country is currently 70 million but is declining at the rate of 1% each year. Let y_n be the population after n years. Find a difference equation showing how to compute y_{n+1} from y_n.

19. One thousand dollars is deposited into an account paying 5% interest compounded annually. At the end of each year \$100 is added to the account. Let y_n be the amount in the account after n years. Find a difference equation satisfied by y_n.

20. The population of a certain country is currently 70 million but is declining at the rate of 1% each year, owing to an excess of deaths over births. In addition the country is losing a million people each year due to emigration. Let y_n be the population after n years. Find a difference equation satisfied by y_n.

21. Consider the difference equation $y_{n+1} = y_n + 2$, $y_0 = 1$.

    (a) Generate y_0, y_1, y_2, y_3, y_4.

    (b) Sketch the graph.

    (c) Why cannot formula (3) be used to obtain the solution?

22. Multiply: $(1 + a + a^2)(1 - a)$. (This result is needed in Section 2.)

23. Suppose you take a consumer loan for \$55 at 20% annual interest and pay off \$36 at the end of each year for two years. Compute the balance on the loan immediately after you make the first payment (that is, the amount you would pay if you wanted to settle the account at the beginning of the second year).

24. Refer to Exercise 23. Set up the difference equation for y_n, the balance after n years.

SOLUTIONS TO PRACTICE PROBLEMS 1

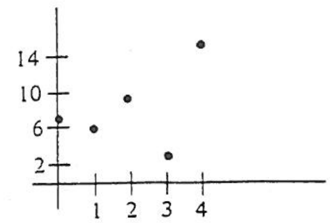
1. (a) $y_0 = 7.5.$

$y_1 = -2(7.5) + 21 = -15 + 21 = 6.$

$y_2 = -2(6) + 21 = -12 + 21 = 9.$

$y_3 = -2(9) + 21 = -18 + 21 = 3.$

$y_4 = -2(3) + 21 = -6 + 21 = 15.$



(b) We graph the points (0, 7.5), (1, 6), (2, 9), (3, 3), (4, 15). (In order to accommodate the points, we use a different scale for each axis.)

(c) $y_n = b/(1 - a) + (y_0 - b/(1 - a))a^n.$

Since $a = -2$ and $b = 21$, $\dfrac{b}{1-a} = \dfrac{21}{1-(-2)} = \dfrac{21}{3} = 7.$

$y_n = 7 + (7.5 - 7)(-2)^n = 7 + \frac{1}{2}(-2)^n.$

2. $y_n = 7 + \frac{1}{2}(-2)^n.$

$y_0 = 7 + \frac{1}{2}(-2)^0 = 7 + \frac{1}{2} = 7.5.$ (Here we have used the fact that any number raised to the zeroth power is 1.)

$y_1 = 7 + \frac{1}{2}(-2)^1 = 7 + \frac{1}{2}(-2) = 7 - 1 = 6.$

$y_2 = 7 + \frac{1}{2}(-2)^2 = 7 + \frac{1}{2}(4) = 7 + 2 = 9.$

10.2. Introduction to Difference Equations, II

The main result of the preceding section is:

> The difference equation $y_{n+1} = ay_n + b$, with $a \neq 1$, has solution
> $$y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n. \qquad (1)$$

When $a = 1$, the difference equation $y_{n+1} = a \cdot y_n + b$ is $y_{n+1} = 1 \cdot y_n + b$ or just $y_{n+1} = y_n + b$. In this case the solution is given by a formula different from (1):

> The difference equation $y_{n+1} = y_n + b$ has the solution
> $$y_n = y_0 + bn. \qquad (2)$$

EXAMPLE 1 (a) Solve the difference equation $y_{n+1} = y_n + 2$, $y_0 = 3$.

(b) Find y_{100}.

Solution (a) Since $a = 1$, the solution is given by (2). Here $b = 2$, $y_0 = 3$. Therefore,

$$y_n = 3 + 2n.$$

(b) $y_{100} = 3 + 2(100) = 203.$

Note: Without (2), determining y_{100} would require 100 computations.

Both (1) and (2) are derived at the end of this section. Before deriving them, however, let us see what they tell us about simple interest, compound interest, and consumer loans.

Simple Interest Suppose that a certain amount of money is deposited in a savings account. If interest is paid only on the initial deposit (and not on accumulated interest), then the interest is called *simple*. For example, if \$40 is deposited at 6% simple interest, then each year the account earns .06(40) or \$2.40. So the bank balance accumulates as follows:

Year	*Amount*	*Interest*
0	\$40	\$2.40
1	42.40	2.40
2	44.80	2.40
3	47.20	

EXAMPLE 2 (a) Find a formula for y_n, the amount in the account above at the end of n years.

(b) Find the amount at the end of 10 years.

Solution Let y_n = the amount at the end of n years. So $y_0 = 40$. Moreover,

$$\begin{array}{ccccc}[\text{amount at end of } n+1 \text{ years}] & = & [\text{amount at end of } n \text{ years}] & + & [\text{interest}]. \\ y_{n+1} & = & y_n & + & 2.40\end{array}$$

This difference equation has $a = 1$, $b = 2.40$, so from formula (2)

$$\begin{aligned} y_n &= y_0 + bn \\ &= 40 + (2.40)n. \end{aligned}$$

This is the desired formula.

(b) $y_{10} = 40 + (2.40)10 = 40 + 24.00 = \$64.$

Compound Interest When interest is calculated on the current amount in the account (instead of on the amount initially deposited), the interest is called *compound.* The interest discussed in Section 1 was compound, being computed on the balance at the end of each year. Such interest is called *annual* compound interest. Often interest is compounded more than once a year. For example, interest might be stated as 6% compounded *semiannually.* This means that interest is computed every six months, with 3% given for each six-month period. At the end of each such period the interest is added to the balance, which is then used to compute the interest for the next six-month period. Similarly, 6% *interest compounded quarterly* means .06/4 or .015 interest four times a year. And 6% interest compounded six times a year means .06/6 or .01 interest each interest period of two months. Or, in general, if 6% interest is compounded k times a year, then $.06/k$ interest is earned k times a year. This illustrates the following general principle:

> If interest is at a yearly rate r and is compounded k times per year, then the interest rate per period (denoted i) is $i = r/k$.

EXAMPLE 3 Suppose that the interest rate is 6% compounded monthly. Find a formula for the amount after n months.

Solution Here $r = .06$, $k = 12$, so that the monthly interest rate is $i = .005$. Let y_n denote the amount after n months. Then, reasoning as in the preceding section,

$$\begin{aligned} y_{n+1} &= y_n + .005y_n \\ &= 1.005y_n. \end{aligned}$$

The solution of this difference equation is obtained as follows:

$$\frac{b}{1-a} = 0$$

$$\begin{aligned} y_n &= 0 + (y_0 - 0)(1.005)^n \\ &= y_0(1.005)^n. \end{aligned}$$

EXAMPLE 4 Suppose that interest is computed at the rate i per interest period. Find a general formula for the balance y_n after n periods under (a) simple interest, (b) compound interest.

Solution (a) With simple interest, the interest each period is just i times the initial amount—that is, $i \cdot y_0$. Therefore,

$$y_{n+1} = y_n + iy_0.$$

Apply (2) with $b = iy_0$. Then

$$y_n = y_0 + (iy_0)n.$$

(b) With compound interest, the interest each period is i times the current balance or $i \cdot y_n$. Therefore,

$$y_{n+1} = y_n + iy_n$$
$$= (1 + i)y_n.$$

Here $b/(1 - a) = 0$, so that

$$y_n = 0 + (y_0 - 0)(1 + i)^n$$
$$= y_0(1 + i)^n.$$

Summarizing:

> **If y_0 dollars is deposited at interest rate i per period, then the amount after n interest periods is:**
>
> **Simple interest:** $\quad y_n = y_0 + (iy_0)n.$ (3)
>
> **Compound interest:** $\quad y_n = y_0(1 + i)^n.$ (4)

EXAMPLE 5 How much money will you have after seven years if you deposit \$40 at 8% interest compounded quarterly?

Solution Apply formula (4). Here a period is a quarter or three months. In seven years there are $7 \cdot 4 = 28$ periods. The interest rate per period is

$$i = \frac{.08}{4} = .02.$$

The amount after 28 periods is

$$y_{28} = 40(1.02)^{28}$$
$$= \$69.64.$$

[To compute $(1.02)^{28}$ we used a calculator. The answer could have been left in the form $40(1.02)^{28}$.]

Consumer Loans It is common for people to buy cars or appliances "on time." Basically, they are borrowing money from the dealer and repaying it (with interest) with several equal payments until the loan is paid off. Each time period, part of the payment goes toward paying off the interest and part toward reducing the balance of the loan. A consumer loan used to purchase a house is called a *mortgage*.

EXAMPLE 6 Suppose that a consumer loan of \$2400 carries an interest rate of 12% compounded annually and a yearly payment of \$1000.

(a) Write down a difference equation for y_n, the balance owed after n years.

(b) Compute the balances after 1, 2, and 3 years.

Solution (a) At the end of each year the new balance is computed as follows:

$$[\text{new balance}] = [\text{previous balance}] + [\text{interest}] - [\text{payment}].$$

Since the interest is compound, it is computed on the previous balance:

$$y_{n+1} = y_n + .12y_n - 1000.$$

Thus,

$$y_{n+1} = 1.12y_n - 1000, \qquad y_0 = 2400,$$

the desired difference equation.

(b) $y_1 = 1.12y_0 - 1000 = (1.12)(2400) - 1000 = \$1688$

$y_2 = 1.12y_1 - 1000 = (1.12)(1688) - 1000 = 891$

$y_3 = 1.12y_2 - 1000 = (1.12)(891) - 1000 \approx 0.$

Although savings accounts using simple interest are practically unheard of, consumer loans are often computed using simple interest. Under simple interest, interest is paid on the entire amount of the initial loan, not just on the outstanding balance.

EXAMPLE 7 Rework the preceding example, except assume that the interest is now simple.

Solution (a) As before,

$$[\text{new balance}] = [\text{previous balance}] + [\text{interest}] - [\text{payment}].$$

Now, however, since the interest is simple, it is computed on the original loan y_0. So

$$y_{n+1} = y_n + .12y_0 - 1000, \qquad y_0 = 2400.$$

(b) $y_1 = y_0 + .12y_0 - 1000 = \$1688$

$y_2 = y_1 + .12y_0 - 1000 = 976$

$y_3 = y_2 + .12y_0 - 1000 = 264.$

Remark: Note that after three years the loan of Example 7 is not yet paid off, whereas the loan of Example 6 is. Therefore the loan at 12% simple interest is more expensive than the one at 12% compound interest. Actually, a 12% simple interest loan is equivalent to a 16% compound interest loan. It was at one time a common practice to advertise loans in terms of simple interest to make the interest rate seem cheaper. But the Federal Truth in Lending Law now requires that all loans be stated in terms of their equivalent compound interest rate.

Verification of Formulas (1) and (2) The derivation of formula (1) depends on the formula for the sum of the powers of a number. If $a \neq 1$, then

$$1 + a = \frac{1 - a^2}{1 - a}$$

$$1 + a + a^2 = \frac{1 - a^3}{1 - a}$$

$$1 + a + a^2 + a^3 = \frac{1 - a^4}{1 - a}.$$

In general, for any positive integer r the sum of the first r powers of a is given by $1 - a^{r+1}$ divided by $1 - a$. Symbolically:

$$\boxed{\begin{array}{l}\text{If } a \neq 1, \text{ then} \\ \qquad 1 + a + a^2 + \ldots + a^r = \dfrac{1 - a^{r+1}}{1 - a}, \\ \text{where } r \text{ is any positive integer.}\end{array}} \tag{5}$$

Note that the condition $a \neq 1$ is essential to avoid dividing by zero on the right.

Illustration of Formula (5) When $a = .9$ and $r = 2$, then

$$1 + a + a^2 = 1 + .9 + .9^2 = 1 + .9 + .81 = 2.71$$

$$\frac{1 - a^3}{1 - a} = \frac{1 - .9^3}{1 - .9} = \frac{1 - .729}{1 - .9} = \frac{.271}{.1} = 2.71.$$

Verification of Formula (5) Form the product

$$(1 + a + a^2 + \ldots + a^r)(1 - a).$$

Multiplying the product out, we get

$$\begin{array}{l} 1 + a + a^2 + \ldots + a^r \\ \underline{1 - a \qquad\qquad\qquad\quad} \\ 1 + a + a^2 + \ldots + a^r \\ \underline{\quad - a - a^2 - a^3 + \ldots - a^{r+1}} \\ 1 + 0 + 0 \ + \ldots + 0 \ - a^{r+1}. \end{array}$$

Thus,

$$(1 + a + a^2 + \ldots + a^r)(1 - a) = 1 - a^{r+1}.$$

If $a \neq 1$, then $1 - a \neq 0$, and we may divide both sides by $1 - a$ to get formula (5).

The form in which formula (5) will be needed is as follows. Let n be any integer >1. Then

$$1 + a + a^2 + \ldots + a^{n-1} = \frac{1 - a^n}{1 - a}. \tag{6}$$

This formula is obtained by replacing r in formula (5) by $n - 1$.

Verification of formula (1)

From the difference equation $y_{n+1} = ay_n + b$ we get

$$\begin{aligned} y_1 &= ay_0 + b \\ y_2 &= ay_1 + b = a(ay_0 + b) + b \\ &= a^2y_0 + ab + b \\ &= a^2y_0 + b(1 + a) \\ y_3 &= ay_2 + b = a(a^2y_0 + ab + b) + b \\ &= a^3y_0 + a^2b + ab + b \\ &= a^3y_0 + b(1 + a + a^2). \end{aligned}$$

The pattern which clearly develops is

$$y_n = a^ny_0 + b(1 + a + a^2 + \ldots + a^{n-1}). \tag{7}$$

By formula (6)

$$\begin{aligned} y_n &= a^ny_0 + b \cdot \frac{1 - a^n}{1 - a} \\ &= a^ny_0 + b \cdot \frac{1}{1 - a} - b \cdot \frac{a^n}{1 - a} \\ &= \frac{b}{1 - a} + \left(y_0 - \frac{b}{1 - a}\right)a^n, \end{aligned}$$

which is formula (1).

Verification of formula (2)

Formula (1) gives the solution of the difference equation $y_{n+1} = ay_n + b$ in the case $a \neq 1$. The reasoning above also gives the solution in case $a = 1$—namely equation (7) holds for any value of a. In particular, for $a = 1$ equation (7) reads

$$\begin{aligned} y_n &= a^ny_0 + b(1 + a + a^2 + \ldots + a^{n-1}) \\ &= 1^ny_0 + b(1 + 1 + 1^2 + \ldots + 1^{n-1}) \\ &= y_0 + b\underbrace{(1 + 1 + 1 + \ldots + 1)}_{n \text{ ones}} \\ &= y_0 + bn. \end{aligned}$$

Thus, we have derived formula (2).

PRACTICE PROBLEMS 2

1. Solve the following difference equations.

   (a) $y_{n+1} = -.2y_n + 24, \quad y_0 = 25.$ (b) $y_{n+1} = y_n - 3, \quad y_0 = 7.$

2. In 1626 Peter Minuit, the first director-general of New Netherlands province, purchased Manhattan Island for trinkets and cloth valued at about \$24. Suppose that this money had been invested at 7% interest compounded quarterly. How much would it have been worth by the United States bicentennial year, 1976?

EXERCISES 2

1. Solve the difference equation $y_{n+1} = y_n + 5$, $y_0 = 1$.

2. Solve the difference equation $y_{n+1} = y_n - 2$, $y_0 = 50$.

In Exercises 3–8 find an expression for the amount of money in the bank after five years where the initial deposit is \$80 and the annual interest rate is as given.

3. 9% compounded monthly.
4. 4% simple interest.
5. 100% compounded daily.
6. 8% compounded quarterly.
7. 7% simple interest.
8. 12% compounded semiannually.

9. Find the general formula for the amount of money accumulated after t years when A dollars is invested at annual interest rate r compounded k times per year.

10. Determine the amount of money accumulated after one year when \$1 is deposited at 40% interest compounded

(a) annually (b) semiannually (c) quarterly.

11. For the difference equation $y_{n+1} = 2y_n - 10$ generate y_0, y_1, y_2, y_3 and draw the graph corresponding to the initial condition:

(a) $y_0 = 10$ (b) $y_0 = 11$ (c) $y_0 = 9$.

12. For the difference equation $y_{n+1} = \frac{1}{2}y_n + 5$ generate y_0, y_1, y_2, y_3 and draw the graph corresponding to the initial condition:

(a) $y_0 = 10$ (b) $y_0 = 18$ (c) $y_0 = 2$.

In Exercises 13–16 solve the difference equation and, by inspection, determine the long-run behavior of the terms (that is, the behavior of the terms as n gets large).

13. $y_{n+1} = .4y_n + 3$, $y_0 = 7$.
14. $y_{n+1} = 3y_n - 12$, $y_0 = 10$.
15. $y_{n+1} = -5y_n$ $y_0 = 2$.
16. $y_{n+1} = -.7y_n + 3.4$, $y_0 = 3$.

17. A bank loan of \$38,900 at 9% interest compounded monthly is made in order to buy a house and is paid off at the rate of \$350 per month for 20 years. (Such a loan is called a *mortgage*.) The balance at any time is the amount still owed on the loan, that is, the amount which would have to be paid out to repay the loan all at once at that time. Find the difference equation for y_n, the balance after n months.

18. Refer to Exercise 17. Express in mathematical notation the fact that the loan is paid off after 20 years.

19. A house is purchased for \$50,000 and depreciated over a 25-year period. Let y_n be the (undepreciated) value of the house after n years. Determine and solve the difference equation for y_n assuming straight-line depreciation (that is, each year the house depreciates by one twenty-fifth of its original value).

20. Refer to Exercise 19. Determine and solve the difference equation for y_n assuming the double-declining balance method of depreciation (that is, each year the house depreciates by two twenty-fifths of its value at the beginning of that year).

SOLUTIONS TO PRACTICE PROBLEMS 2

1. (a) Since $a = -.2 \neq 1$, use formula (1).

$$\frac{b}{1-a} = \frac{24}{1-(-.2)} = \frac{24}{1.2} = \frac{240}{12} = 20,$$

$$y_n = 20 + (25 - 20)(-.2)^n = 20 + 5(-.2)^n.$$

(b) Since $a = 1$, use formula (2).

$$y_n = 7 + (-3)n = 7 - 3n.$$

2. Since interest is compounded quarterly, the interest per period is $.07/4 = .0175$. Three hundred and fifty years consists of $4(350) = 1400$ interest periods. Therefore by (4), the amount accumulated is

$$y_{1400} = 24(1.0175)^{1400}.$$

(This amount is approximately 850 billion dollars, which is more than Manhattan Island was worth in 1976.)

10.3. Graphing Difference Equations

In this section we introduce a method for sketching the graph of the difference equation $y_{n+1} = ay_n + b$ (with initial value y_0) directly from the three numbers a, b, and y_0. As we shall see, the graphs arising from difference equations can be completely described by two characteristics—vertical direction and long-run behavior. To begin we introduce some vocabulary to describe graphs.

The *vertical direction* of a graph refers to the up-and-down motion of successive terms. A graph *increases* if it rises when read from left to right—that is, if the terms get successively larger. A graph *decreases* if it falls when read from left to right—that is, if the terms get successively smaller. Figure 1 shows the graphs of two difference equations. Both graphs increase. Figure 2 shows two examples of

FIGURE 1 Increasing graphs.

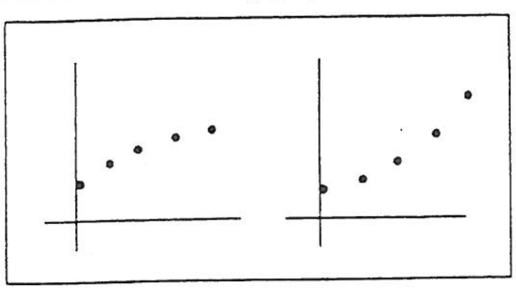
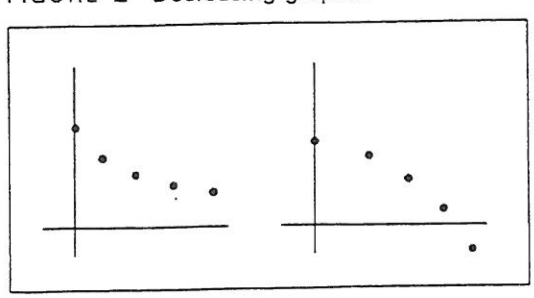


FIGURE 2 Decreasing graphs.



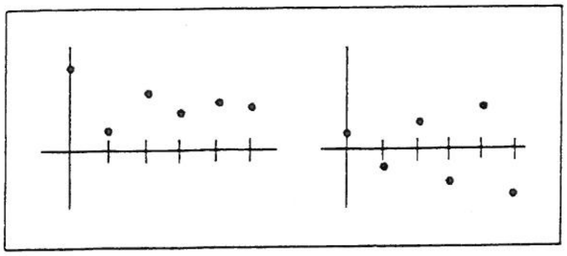


FIGURE 3
Oscillating graphs.

decreasing graphs. A graph which is either increasing or decreasing is called *monotonic*. That is, a graph is monotonic if it always heads in one direction—up or down.

The extreme opposite of a monotonic graph is one that changes its direction with every term. Such a graph is called *oscillating*. Figure 3 shows two examples of oscillating graphs. A difference equation having an oscillating graph has terms y_n that alternately increase and decrease.

In addition to the monotonic and oscillating graphs, there are the *constant* graphs, which always remain at the same height. That is, all the terms y_n are the same. A constant graph is illustrated in Fig. 4.

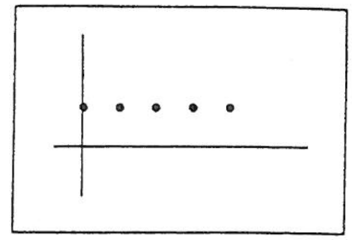


FIGURE 4 Constant graph.

One of the main results of this section is that the graph of a difference equation $y_{n+1} = ay_n + b$ is always either monotonic, oscillating, or constant. This threefold classification gives us all the possibilities for the vertical direction of the graph.

Long-run behavior refers to the eventual behavior of the graph. Most graphs of difference equations exhibit one of two types of long-run behavior. Some approach a horizontal line and are said to be *asymptotic* to the line. Some go indefinitely high or indefinitely low and are said to be *unbounded*. These phenomena are illustrated in Figs. 5 and 6.

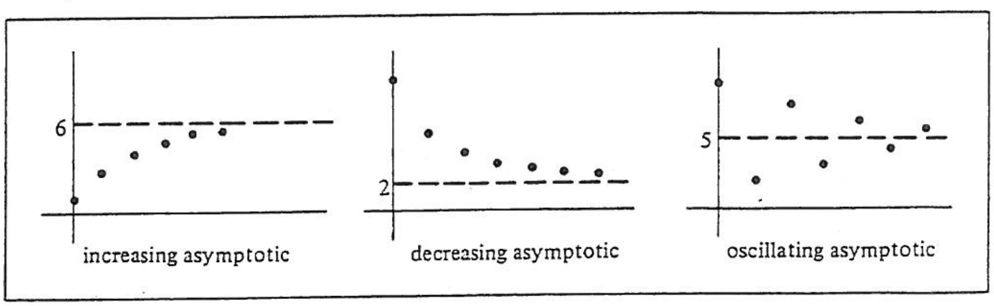
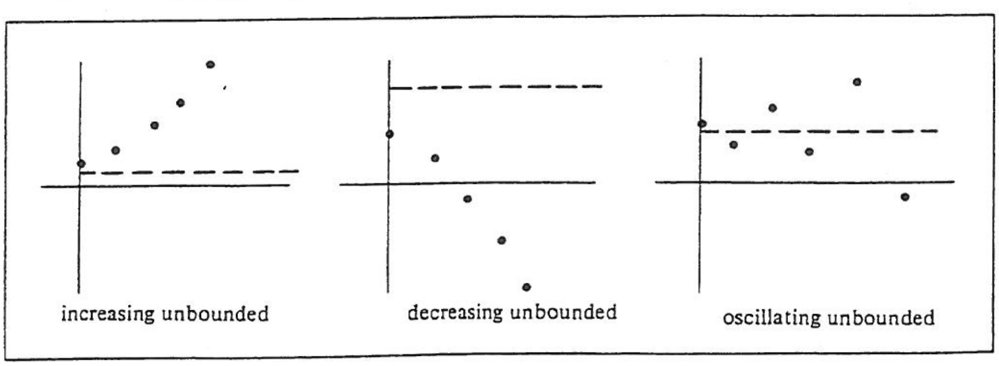


FIGURE 5 Asymptotic graphs.

FIGURE 6 Unbounded graphs.



The dashed horizontal lines in Figs. 5 and 6 each have the equation $y = b/(1 - a)$. Note that in Fig. 5 the terms move steadily closer to the dashed line and in Fig. 6 they move steadily further away from the dashed line. In the first case we say that the terms are *attracted* to the dashed line and in the second case we say that they are *repelled* by it.

Constant Graphs The general formula for the nth term of a difference equation where $a \neq 1$ is

$$y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n.$$

It is clear that as n varies so does a^n, and this makes y_n vary with n—unless, of course, the coefficient of a^n is 0, in which case the graph is constant. Thus, we have the following result:

> The graph of $y_{n+1} = ay_n + b$ $(a \neq 1)$ is constant if $y_0 - b/(1 - a) = 0$; that is, if $y_0 = b/(1 - a)$.

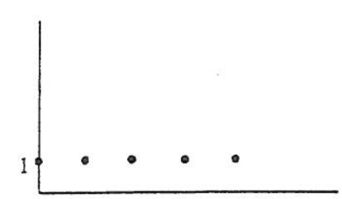
Thus, when the graph starts out on the line $y = b/(1 - a)$, it stays on the line.

EXAMPLE 1 Sketch the graph of the difference equation $y_{n+1} = 2y_n - 1$, $y_0 = 1$.

Solution First compute $b/(1 - a)$:

$$\frac{b}{1-a} = \frac{-1}{1-2} = 1.$$

But $y_0 = 1$. So $b/(1 - a)$ and y_0 are the same and the graph is constant, always equal to 1 (Fig. 7).



$y_{n+1} = 2y_n - 1, y_0 = 1$

FIGURE 7

Throughout this section assume that $a \neq \pm 1$. Then the graph of a given difference equation either is constant or is one of the types shown in Figs. 5 and 6. We can determine the nature of the graph by looking at the coefficients a and b and the initial value y_0. We have seen how constant graphs are handled. Thus, assume that we are dealing with a nonconstant graph—that is, the graph of a difference equation for which a^n actually affects the formula for y_n. Monotonic graphs may be differentiated from oscillating graphs by the following test.

> *Test 1:* If $a > 0$, then the graph of $y_{n+1} = ay_n + b$ is monotonic. If $a < 0$, then the graph is oscillating.

The next two examples provide a convincing argument for Test 1.

EXAMPLE 2 Discuss the vertical direction of the graph of $y_{n+1} = -.8y_n + 9$, $y_0 = 50$.

Solution The formula for y_n yields

$$y_n = 5 + 45(-.8)^n.$$

Note that the term $(-.8)^n$ is alternately positive and negative, since any negative number to an even power is positive and any negative number to an odd power is negative. Therefore, the expression $45(-.8)^n$ is alternately positive and negative. So $y_n = 5 + 45(-.8)^n$ is computed by alternately adding and subtracting something from 5. Thus, y_n oscillates around 5. In this example $a = -.8$ (a negative number), so that the behavior just observed is consistent with that predicted by Test 1. The graph is sketched in Fig. 8. Note that in this case 5 is just $b/(1-a)$, so the graph oscillates about the line $y = b/(1-a)$.

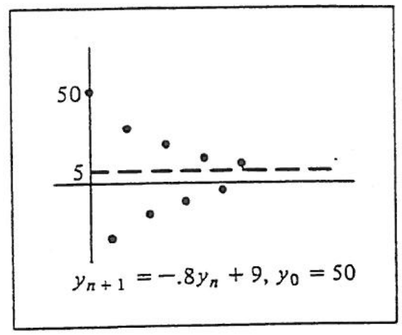


FIGURE 8

The reasoning of the preceding example works whenever $a < 0$. The oscillation of a^n from positive to negative and back forces the term $(y_0 - [b/(1-a)])a^n$ to swing up and back from positive to negative. So the value of

$$y_n = \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^n$$

swings up and back, above and below $b/(1-a)$. In other words, the graph oscillates about the line $y = b/(1-a)$.

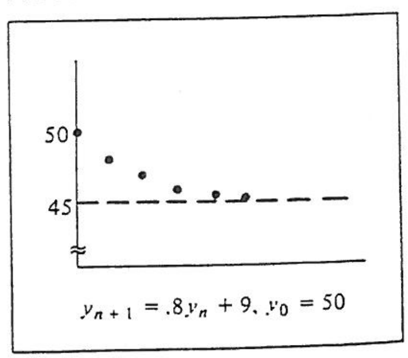
EXAMPLE 3 Discuss the vertical direction of the graph of $y_{n+1} = .8y_n + 9$, $y_0 = 50$.

Solution In this case $b/(1-a) = 45$, and the formula for y_n gives

$$y_n = 45 + 5(.8)^n.$$

The expression $5(.8)^n$ is always positive. As n gets larger, $5(.8)^n$ gets smaller, so that y_n decreases to 45. That is, the graph is steadily decreasing to $y = 45$ (Fig. 9). Here $a = .8 > 0$, and so Test 1 correctly predicts that the graph is monotonic.

FIGURE 9



It is possible to determine whether a graph is asymptotic or unbounded, using the following result.

> *Test 2:* If $|a| < 1$, then the graph of $y_{n+1} = ay_n + b$ is asymptotic to the line $y = b/(1 - a)$. If $|a| > 1$, then the graph is unbounded and moves away from the line $y = b/(1 - a)$.

Let us examine some difference equations in light of Test 2.

EXAMPLE 4 Discuss the graphs of the difference equation $y_{n+1} = .2y_n + 4.8$, with $y_0 = 1$ and with $y_0 = 11$.

Solution The graphs are shown in Fig. 10. Since $a = .2$, which has absolute value less than 1, Test 2 correctly predicts that each graph is asymptotic to the line $y = b/(1 - a) = 4.8/(1 - .2) = 6$. When the initial value is less than 6, the graph increases and moves toward the line $y = 6$. When the initial value is greater than 6, the graph decreases toward the line $y = 6$. In each case the graph is *attracted* to the line $y = 6$.

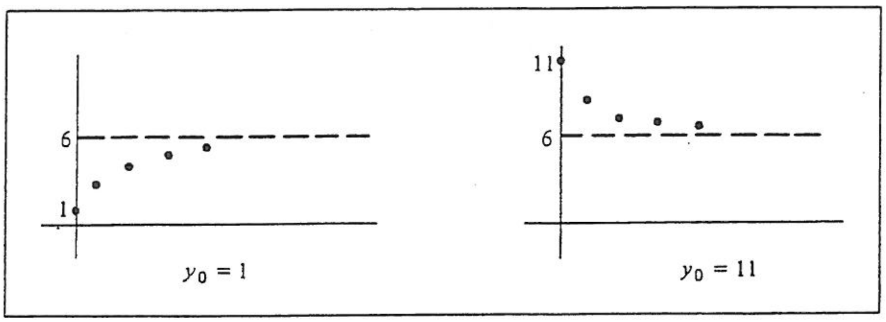


FIGURE 10

EXAMPLE 5 Apply Test 2 to the difference equation $y_{n+1} = -.8y_n + 9$, $y_0 = 50$.

Solution $|a| = |-.8| = .8 < 1$. So by Test 2 the graph is asymptotic to the line $y = b/(1 - a) = 5$. This agrees with the graph as drawn in Fig. 8.

EXAMPLE 6 Discuss the graphs of $y_{n+1} = 1.4y_n - 8$, with $y_0 > 20$ and with $y_0 < 20$.

Solution The graphs are shown in Fig. 11. Since $a = 1.4 > 0$, Test 1 predicts that the graphs

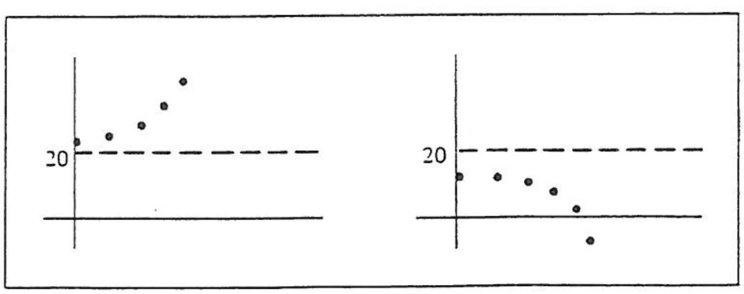


FIGURE 11

are monotonic. Since $|a| = |1.4| = 1.4 > 1$, Test 2 predicts that the graphs are unbounded. Here

$$\frac{b}{1-a} = \frac{-8}{1-(1.4)} = \frac{-8}{-.4} = 20.$$

Both graphs move away from the line $y = 20$ as if being *repelled* by a force.

EXAMPLE 7 Discuss the graph of the difference equation $y_{n+1} = -2y_n + 60$, where $y_0 > 20$.

Solution A graph is drawn in Fig. 12. Since $a = -2 < 0$, Test 1 predicts that the graph is oscillating. Since $|a| = |-2| = 2 > 1$, Test 2 predicts that the graph is unbounded. Here

$$\frac{b}{1-a} = \frac{60}{1-(-2)} = \frac{60}{3} = 20.$$

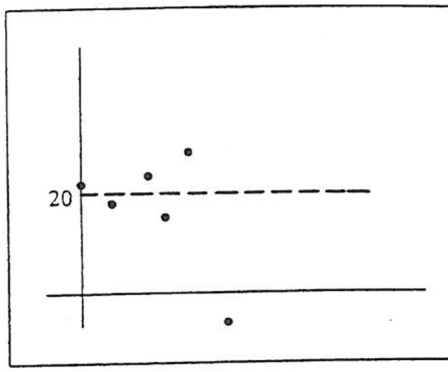


FIGURE 12

Notice that successive points move further away from $y = 20$—that is, they are repelled by the line.

Verification of Test 2 The key to verifying Test 2 is to make the following two observations:

1. If $|a| < 1$, then the powers $a, a^2, a^3, a^4, \ldots,$ become successively smaller and approach 0. (For example, if $a = .4$, then this sequence of powers is .4, .16, .064, .0256, ...)

2. If $|a| > 1$, then the powers $a, a^2, a^3, a^4, \ldots,$ become unbounded; that is, they become arbitrarily large. (For example, if $a = 3$, then this sequence is 3, 9, 27, 81, ...)

To verify Test 2 look at the formula for y_n:

$$y_n = \frac{b}{1-a} + \boxed{\left(y_0 - \frac{b}{1-a}\right)a^n}.$$

If $|a| < 1$, then the powers of a get smaller and smaller, and so the boxed term approaches 0. That is, y_n approaches $b/(1-a)$ and the graph is asymptotic to $y = b/(1-a)$. If $|a| > 1$, the powers of a are unbounded, so that the boxed term becomes arbitrarily large in magnitude. Thus, y_n is unbounded, and so is the graph.

In the case where $|a| < 1$ the graphs (whether oscillating or monotone) are *attracted* steadily to the line $y = b/(1-a)$. In the case where $|a| > 1$ the graphs are *repelled* by the line $y = b/(1-a)$. Thinking of graphs as being attracted or repelled helps us in making a rough sketch.

Sign of a:	Positive	Monotonic
	Negative	Oscillating
Size of a:	$\lvert a \rvert < 1$	Attract
	$\lvert a \rvert > 1$	Repel

Figure 13 shows some of the graphs already examined with the appropriate descriptive words labeling the line $y = b/(1 - a)$.

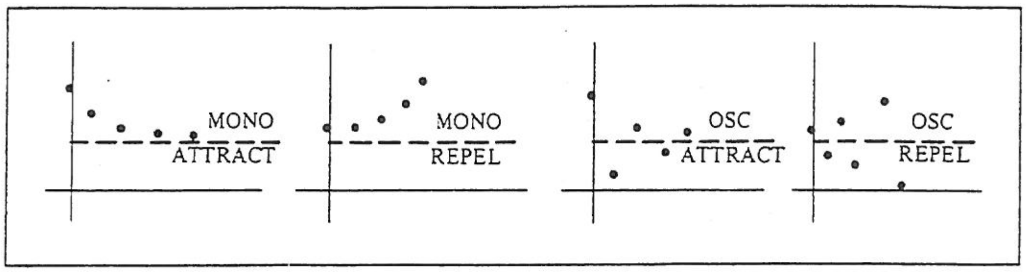


FIGURE 13

Based on Tests 1 and 2 and the discussion of constant graphs, we can state a procedure for making a rough sketch of a graph without solving or generating terms.

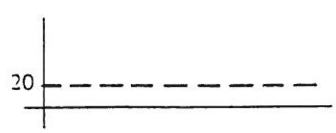
Sketching the graph of $y_{n+1} = ay_n + b$, $a \neq 0, \pm 1$.

1. Draw the line $y = b/(1 - a)$ as a dashed line.
2. Plot y_0. If y_0 is on the line $y = b/(1 - a)$, the graph is constant and the procedure terminates.
3. If a is positive, write MONO, since the graph is then monotonic. If a is negative, write OSC, since the graph is then oscillating.
4. If $|a| < 1$, write ATTRACT, since the graph is attracted to the line $y = b/(1 - a)$. If $|a| > 1$, write REPEL, since the graph is repelled from the line.
5. Use all the information to sketch the graph.

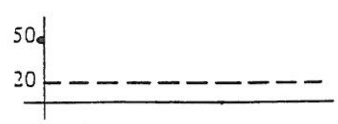
EXAMPLE 8 Sketch the graph of $y_{n+1} = .6y_n + 8$, $y_0 = 50$.

Solution

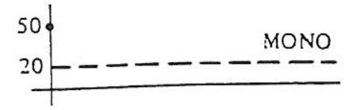
1. $\dfrac{b}{1 - a} = \dfrac{8}{1 - (.6)} = \dfrac{8}{.4} = 20$. So draw the line $y = 20$.



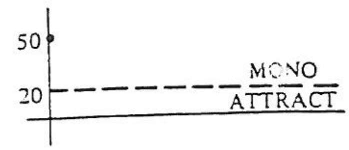
2. $y_0 = 50$, which is not on the line $y = 20$. So the graph is not constant.



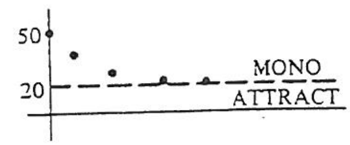
3. $a = .6$, which is positive. Write MONO above the dotted line.



4. $|a| = |.6| = .6 < 1$. Write ATTRACT below the dotted line.



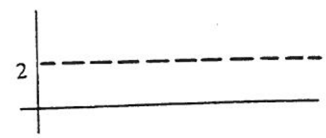
5. The information above tells us to start at 50 and move monotonically toward the line $y = 20$.



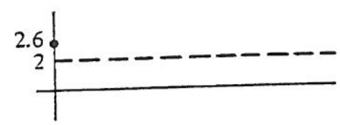
EXAMPLE 9 Sketch the graph of $y_{n+1} = -1.5y_n + 5$, $y_0 = 2.6$.

Solution

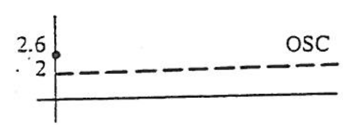
1. $\dfrac{b}{1-a} = \dfrac{5}{1-(-1.5)} = \dfrac{5}{2.5} = 2$. So draw the line $y = 2$.



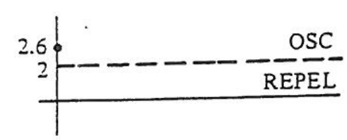
2. $y_0 = 2.6$, which is not on the line $y = 2$. So the graph is not constant.



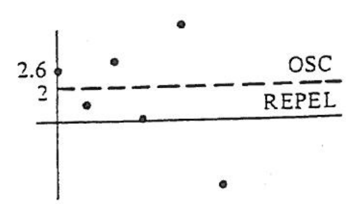
3. $a = -1.5$, which is negative. Write OSC above the dotted line.



4. $|a| = |-1.5| = 1.5 > 1$. Write REPEL below the dotted line.



5. The above information says that the graph begins at 2.6, oscillates about and is steadily repelled by the line $y = 2$.



The procedure above does not give an exact graph, but it shows the nature of the graph. This is exactly what is needed in many applications.

EXAMPLE 10 Suppose that the yearly interest rate on a mortgage is 9% compounded monthly and that you can afford to make payments of $300 per month. How much can you afford to borrow?

Solution Let i = the monthly interest rate, R = the monthly payment, and y_n = the balance after n months. y_0 = the initial amount of the loan. Then the balance after $n + 1$ months equals the balance after n months plus the interest on that balance minus the monthly payment. That is,

$$\begin{aligned} y_{n+1} &= y_n + iy_n - R \\ &= (1 + i)y_n - R. \end{aligned}$$

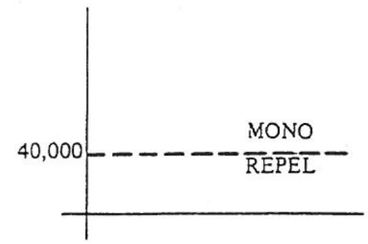
In this particular example $i = .09/12 = .0075$ and $R = 300$, so the difference equation reads

$$y_{n+1} = 1.0075y_n - 300.$$

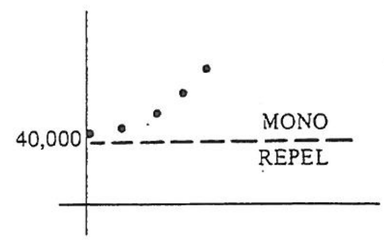
Apply our graph-sketching technique to this difference equation. Here

$$\frac{b}{1-a} = \frac{-300}{1-(1.0075)} = \frac{-300}{-.0075} = 40{,}000.$$

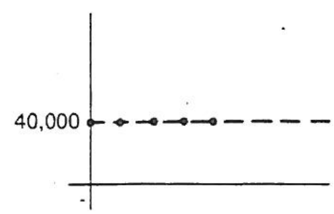
Since $a = 1.0075$ is positive and $|a| = 1.0075 > 1$, the words MONO and REPEL describe the graphs.



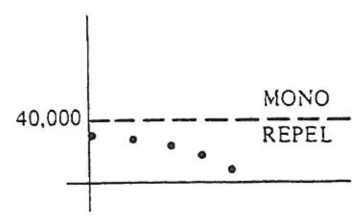
Let us see what happens for various initial values. If $y_0 > 40{,}000$, then the balance increases indefinitely.



If $y_0 = 40{,}000$, then the balance will always be 40,000.



If $y_0 < 40{,}000$, then the balance decreases steadily and eventually reaches 0, at which time the mortgage is paid off.



Therefore, the loan must be less than \$40,000 in order for it to be paid off eventually.

PRACTICE PROBLEMS 3

1. A parachutist opens her parachute after reaching a speed of 100 feet per second. Suppose that y_n, the speed n seconds after opening the parachute, satisfies $y_{n+1} = .1y_n + 14.4$. Sketch the graph of y_n.

2. Rework Problem 1 if the parachute opens when the speed is 10 feet per second.

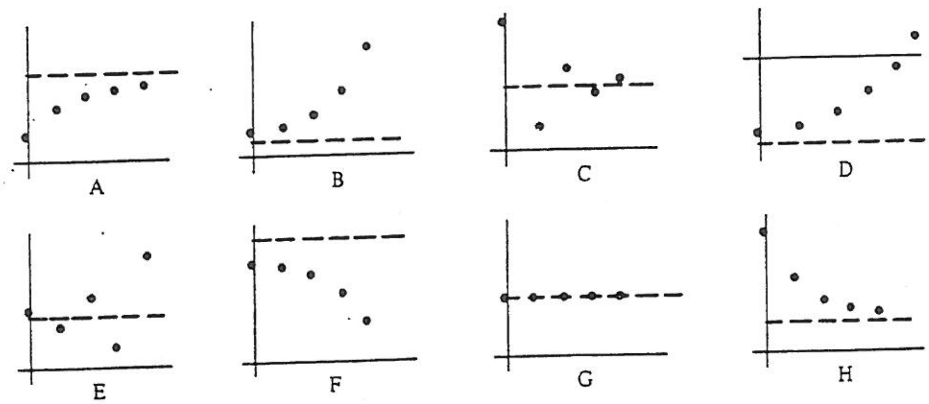
3. Upon retirement, a person deposits a certain amount of money into the bank at 6% interest compounded monthly and withdraws \$100 at the end of each month.

   (a) Set up the difference equation for y_n, the amount in the bank after n months.

   (b) How large must the initial deposit be so that the money will never run out?

EXERCISES 3

Each of the graphs pictured below comes from a difference equation of the form $y_{n+1} = ay_n + b$. In Exercises 1–8 list all graphs which have the stated property.



1. Monotonic.
2. Increasing.
3. Unbounded.
4. Constant.
5. Repelled from $y = b/(1 - a)$.
6. Decreasing.
7. $|a| < 1$.
8. $a < 0$.

In Exercises 9–14 sketch a graph having the given characteristics.

9. $y_0 = 4$, monotonic, repelled from $y = 2$.
10. $y_0 = 5$, monotonic, attracted to $y = -3$.
11. $y_0 = 2$, oscillating, attracted to $y = 6$.
12. $y_0 = 7$, monotonic, repelled from $y = 10$.
13. $y_0 = -2$, monotonic, attracted to $y = 5$.
14. $y_0 = 1$, oscillating, repelled from $y = 2$.

In Exercises 15–20 make a rough sketch of the graph of the difference equation without generating terms or solving the difference equation.

15. $y_{n+1} = 3y_n + 4, \quad y_0 = 0.$

16. $y_{n+1} = .3y_n + 3.5, \quad y_0 = 2.$

17. $y_{n+1} = .5y_n + 3, \quad y_0 = 6.$

18. $y_{n+1} = 4y_n - 18, \quad y_0 = 5.$

19. $y_{n+1} = -2y_n + 12, \quad y_0 = 5.$

20. $y_{n+1} = -.6y_n + 6.4, \quad y_0 = 1.$

21. A particular news item was broadcast regularly on radio and TV. Let y_n be the number of people who had heard the news within n hours after broadcasting began. Sketch the graph of y_n, assuming that it satisfies the difference equation, $y_{n+1} = .7y_n + 3000$, $y_0 = 0$.

22. The radioactive element strontium 90 emits particles and slowly decays. Let y_n be the amount left after n years. Then y_n satisfies the difference equation $y_{n+1} = .98y_n$. Sketch the graph of y_n if initially there is 10 milligrams of strontium 90.

23. Laws of supply and demand cause the price of oats to fluctuate from year to year. Suppose that the current price is \$1.25 per bushel and that the price n years from now, p_n, satisfies the difference equation $p_{n+1} = -.6p_n + 1.6$. (Prices are assumed to have been adjusted for inflation.) Sketch the graph of p_n.

24. Under ideal conditions a bacteria population satisfies the difference equation $y_{n+1} = 1.4y_n$, $y_0 = 1$, where y_n is the size of the population (in millions) after n hours. Sketch a graph which shows the growth of the population.

25. Suppose that the interest rate on a mortgage is 9% compounded monthly. If you can afford to pay \$450 per month, how much money can you borrow?

26. A municipal government can take out a long-term construction loan at 8% interest compounded quarterly. Assuming that it can pay back \$100,000 per quarter, how much money can it borrow?

27. A person makes an initial deposit into a savings account paying 6% interest compounded annually. He plans to withdraw \$120 at the end of each year.

    (a) Find the difference equation for y_n, the amount after n years.

    (b) How large must y_0 be such that the money will not run out?

28. Suppose that a loan of \$10,000 is to be repaid at \$120 per month and that the annual interest rate is 12% compounded monthly. Then the interest for the first month is .01(10,000) or \$100. The \$120 paid at the end of the first month can be thought of as paying the \$100 interest and paying \$20 toward the reduction of the loan. Therefore the balance after one month is \$10,000 − \$20 = \$9980. How much of the \$120 paid at the end of the second month goes for interest and how much is used to reduce the loan? What is the balance after two months?

SOLUTIONS TO PRACTICE PROBLEMS 3

1. $\dfrac{b}{1-a} = \dfrac{14.4}{1-(.1)} = \dfrac{14.4}{.9} = \dfrac{144}{9} = 16.$ $y_0 = 100$, which is greater than $b/(1-a)$. Since $a = .1$ is positive, the terms y_n are monotonic. Since $|a| = .1 < 1$, the terms

are attracted to the line $y = 16$. Now, plot $y_0 = 100$, draw the line $y = 16$, and write in the words MONO and ATTRACT [Fig. 14(a)]. Since the terms are attracted to the line monotonically they must move downward and asymptotically

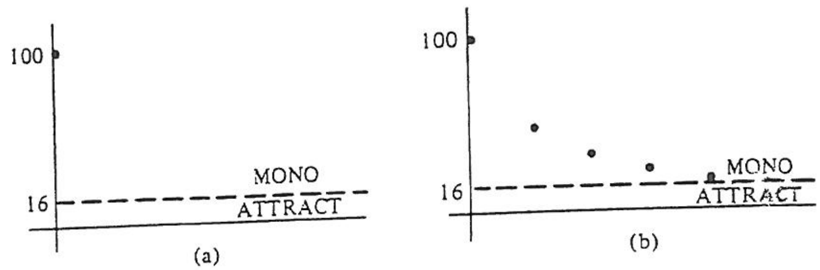
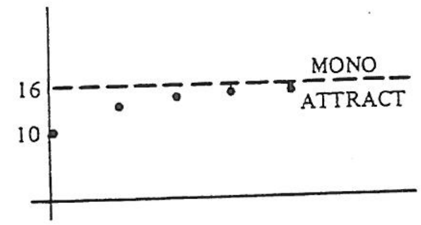


FIGURE 14

approach the line [Fig. 14(b)]. (Notice that the terminal speed, 16 feet per second, does not depend on the speed of the parachutist when the parachute is opened.)

2. Everything is the same as above except that now $y_0 < 16$. The speed increases to a terminal speed of 16 feet per second.

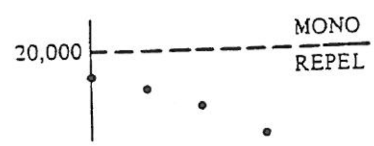


3. (a) $i = .06/12 = .005$.

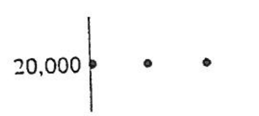
$$\begin{aligned} y_{n+1} &= y_n + (\text{interest}) - (\text{withdrawal}) \\ &= y_n + .005y_n - 100 \\ &= (1.005)y_n - 100. \end{aligned}$$

(b) $\dfrac{b}{1-a} = \dfrac{-100}{1-(1.005)} = \dfrac{-100}{-.005} = 20{,}000$. Since $a = 1.005 > 0$, the graph is monotonic. Since $|a| = 1.005 > 1$, the graph is repelled from the line $y = 20{,}000$.

If $y_0 < 20{,}000$, then the amount of money in the account decreases and eventually runs out.



If $y_0 = 20{,}000$, then the amount of money in the account stays constant at 20,000.



If $y_0 > 20{,}000$, then the amount of money in the account grows steadily and is unbounded.

Answer: $y_0 \geq 20{,}000$.

10.4. Mathematics of Personal Finance

In this section we apply the theory of difference equations to the study of mortgages and annuities.

Mortgages Most families take out bank loans to pay for a new house. Such a loan, called a *mortgage*, is used to purchase the house and then is repaid with interest in monthly installments over a number of years, usually 25 or 30. The monthly payments are computed so that after exactly the correct length of time the unpaid balance* is 0 and the loan is thereby paid off.

Mortgages can be described by difference equations, as follows. Let y_n be the unpaid balance on the mortgage after n months. In particular, y_0 is the initial amount borrowed. Let i denote the monthly interest rate and R the monthly mortgage payment. Then

$$\begin{array}{ccccccc} \begin{bmatrix}\text{balance after}\\ n+1 \text{ months}\end{bmatrix} & = & \begin{bmatrix}\text{balance after}\\ n \text{ months}\end{bmatrix} & + & \begin{bmatrix}\text{interest for}\\ \text{month}\end{bmatrix} & - & [\text{payment}] \\ y_{n+1} & = & y_n & + & iy_n & - & R \end{array}$$

$$y_{n+1} = (1 + i)y_n - R. \tag{1}$$

EXAMPLE 1 Suppose you can afford to pay \$300 per month and the yearly interest rate is 9%, compounded monthly. Exactly how much can you borrow if the mortgage is to be paid off in 30 years?

Solution The monthly interest rate is $.09/12 = .0075$, so equation (1) becomes in this case

$$y_{n+1} = 1.0075y_n - 300.$$

Further, we are given that the mortgage runs for 30 years, or 360 months. Thus,

$$y_{360} = 0. \tag{2}$$

Our problem is to determine y_0, the amount of the loan. From our general theory we first compute $b/(1 - a)$:

$$\frac{b}{1-a} = \frac{-300}{1-(1.0075)} = 40{,}000.$$

The formula for y_{360} is then given by

$$\begin{aligned} y_{360} &= \frac{b}{1-a} + \left(y_0 - \frac{b}{1-a}\right)a^{360} \\ &= 40{,}000 + (y_0 - 40{,}000)(1.0075)^{360}. \end{aligned}$$

---

* The balance after n months is the amount that would have to be paid at that time in order to retire the debt.

Using a calculator, we find that

$$(1.0075)^{360} \approx 14.73.$$

Therefore,

$$y_{360} = 40{,}000 + (y_0 - 40{,}000)(14.73).$$

However, by equation (2), $y_{360} = 0$, so that

$$\begin{aligned} 0 &= 40{,}000 + (y_0 - 40{,}000)(14.73) \\ &= 14.73y_0 - 549{,}200 \\ y_0 &= \frac{549{,}200}{14.73} = 37{,}284.45. \end{aligned}$$

Thus, the initial amount—that is, the amount which can be borrowed—is \$37,284.45.

In the early months almost all of the monthly payment goes toward interest. With each passing month, however, the amount of interest declines and the amount applied to reducing the debt increases. This phenomenon is illustrated for selected months in Table 1.

TABLE 1

	$n = 0$	$n = 1$	$n = 2$	$n = 120$	$n = 240$	$n = 348$
Balance on loan	37284.45	37264.08	37243.56	33343.22	23681.85	3429.00
Interest	279.63	279.48	279.33	250.07	177.61	25.72
Reduction of debt	20.37	20.52	20.67	49.93	122.39	274.28

EXAMPLE 2 Suppose we have a 30-year mortgage for \$7000 at 12% interest, compounded monthly. Find the monthly payment.

Solution Here the monthly interest rate i is just $.12/12 = .01$ and $y_0 = 7000$. Since the mortgage is for 30 years, we have

$$y_{360} = 0.$$

The difference equation for the mortgage is

$$y_{n+1} = 1.01y_n - R \qquad y_0 = 7000.$$

Now $b/(1 - a) = -R/(1 - 1.01) = R/.01 = 100R$. The formula for y_n is

$$y_n = 100R + (7000 - 100R)(1.01)^n.$$

Thus,

$$0 = y_{360} = 100R + (7000 - 100R)(1.01)^{360}.$$

To complete the problem solve this equation for R. Using a calculator, $(1.01)^{360} \approx 35.95$. Thus,

$$\begin{aligned} 100R + (7000 - 100R)(35.95) &= 0 \\ 3{,}495R &= 251{,}650 \\ R &= \$72.00. \end{aligned}$$

Annuities The term annuity has several meanings. For our purposes an annuity is a bank account into which equal sums are deposited at regular intervals, either weekly, monthly, quarterly, or annually. The money draws interest and accumulates for a certain number of years, after which it becomes available to the investor. Annuities are often used to save for a child's college education or to generate funds for retirement.

The growth of money in an annuity can be described by a difference equation. Let y_n = [the amount of money in the annuity after n time periods], i = [the interest rate per time period], D = [the deposit per time period]. Then y_0 = [the amount after 0 time periods] = 0.* Moreover,

$$\begin{aligned} y_{n+1} &= [\text{previous amount}] + [\text{interest}] + [\text{deposit}] \\ &= \qquad y_n \qquad + \quad iy_n \quad + \quad D \\ &= (1+i)y_n + D. \end{aligned}$$

This last equation is the difference equation of the annuity.

EXAMPLE 3 Suppose that \$20 is deposited into an annuity at the end of every quarter-year and that interest is earned at the annual rate of 8%, compounded quarterly. How much money is in the annuity after 10 years?

Solution Since 10 years = 40 quarters, the problem is to compute y_{40}. Now $D = 20$ and $i = .08/4 = .02$. So the difference equation reads

$$y_{n+1} = 1.02y_n + 20.$$

In this case

$$\frac{b}{1-a} = \frac{20}{1-1.02} = -1000.$$

Thus,

$$\begin{aligned} y_n &= -1000 + [0 - (-1000)](1.02)^n \\ &= -1000 + 1000(1.02)^n. \end{aligned}$$

In particular, setting $n = 40$,

$$y_{40} = -1000 + 1000(1.02)^{40}.$$

* One model for the situation $y_0 = 0$ is a payroll savings plan, where a person signs up at time zero and has the first deduction made at the end of the next pay period.

Using a calculator, $(1.02)^{40} \approx 2.21$. Thus,

$$\begin{aligned} y_{40} &= -1000 + 1000(2.21) \\ &= -1000 + 2210 \\ &= 1210. \end{aligned}$$

Thus, after 10 years the account contains \$1210.

EXAMPLE 4 How much money must be deposited at the end of each quarter into an annuity at 8% interest compounded quarterly in order to have \$10,000 after 15 years?

Solution Here D is unknown. But $i = .02$. Also, 15 years = 60 quarters, so that $y_{60} = 10{,}000$. The difference equation in this case reads

$$y_{n+1} = 1.02y_n + D \qquad y_0 = 0.$$

Then

$$\frac{b}{1-a} = \frac{D}{1-1.02} = \frac{D}{-.02} = -50D$$

and

$$\begin{aligned} y_n &= -50D + (0 - (-50D))(1.02)^n \\ &= -50D + 50D(1.02)^n. \end{aligned}$$

Set $n = 60$. Then

$$y_{60} = -50D + 50D(1.02)^{60}.$$

However, from the statement of the problem $y_{60} = 10{,}000$. Thus, we have the equation

$$10{,}000 = -50D + 50D(1.02)^{60},$$

to be solved for D. But $(1.02)^{60} \approx 3.28$, so that

$$\begin{aligned} -50D + 50(3.28)D &= 10{,}000 \\ -50D + 164D &= 10{,}000 \\ 114D &= 10{,}000 \\ D &= 10{,}000/114 \\ &= \$87.72. \end{aligned}$$

PRACTICE PROBLEMS 4

1. Suppose you deposit \$650,000 into a bank account paying 5% interest compounded annually and you withdraw \$50,000 at the end of each year. Find a difference equation for y_n, the amount in the account after n years.

2. Refer to Problem 1. How much money will be in the account after 20 years? [Note: $(1.05)^{20} \approx 2.65$.]

3. Refer to Problem 1. Assume that the money is tax free and that you could earn 5% interest compounded annually. Would you rather have \$650,000 now or \$50,000 a year for 20 years?

EXERCISES 4

In Exercises 1–4 give the difference equation for y_n, the amount (or balance) after n interest periods.

1. A mortgage loan of \$32,500 at 9% interest compounded monthly and having monthly payments of \$261.50.

2. A bank deposit of \$1000 at 6% interest compounded semiannually

3. An annuity for which \$4000 is deposited into an account at 6% interest compounded quarterly and \$200 is added to the account at the end of each quarter.

4. A bank account into which \$20,000 is deposited at 6% interest compounded monthly and \$100 is withdrawn at the end of each month.

5. How much money can you borrow at 12% interest compounded monthly if the loan is to be paid off in monthly installments for 10 years and you can afford to pay \$660 per month? [Note: $(1.01)^{120} \approx 3.3$.]

6. Find the monthly payment on a \$38,000, 25-year mortgage at 12% interest compounded monthly. [Note: $(1.01)^{300} \approx 20$.]

7. Find the amount accumulated after 20 years if, at the end of each year, \$300 is deposited into an account paying 6% interest compounded annually. [Note: $(1.06)^{20} \approx 3.2$.]

8. How much money would you have to deposit at the end of each month into an annuity paying 6% interest compounded monthly in order to have \$6000 after 12 years? [Note: $(1.005)^{144} \approx 2$.]

9. How much money would you have to put into an account initially at 8% interest compounded quarterly in order to have \$6000 after 14 years? [Note: $(1.02)^{56} \approx 3$.]

10. How much money would you have to put into a bank account paying 6% interest compounded monthly in order to be able to withdraw \$150 each month for 30 years? [Note: $(1.005)^{360} \approx 6$.]

11. In order to buy a car, a person borrows \$4000 from the bank at 12% interest compounded monthly. The loan is to be paid off in three years with equal monthly payments. What will the monthly payments be? [Note: $(1.01)^{36} \approx 1.43$.]

12. How much money would you have to deposit at the end of each month into an annuity paying 6% interest compounded monthly in order to have \$1620 after four years? [Note: $(1.005)^{48} \approx 1.27$.]

SOLUTIONS TO PRACTICE PROBLEMS 4

1. [Amount after $n + 1$ years] = [amount after n years] + [interest for $(n + 1)$st year] − [withdrawal at end of year].

$$y_{n+1} = y_n + .05y_n - 50{,}000$$
$$= (1.05)y_n - 50{,}000, \qquad y_0 = 650{,}000.$$

2. Solve the difference equation and set $n = 20$.

$$\frac{b}{1-a} = \frac{-50{,}000}{1-(1.05)} = \frac{-50{,}000}{-.05} = 1{,}000{,}000,$$
$$y_n = 1{,}000{,}000 + (650{,}000 - 1{,}000{,}000)(1.05)^n$$
$$= 1{,}000{,}000 - 350{,}000(1.05)^n,$$
$$y_{20} \approx 1{,}000{,}000 - 350{,}000(2.65) = 72{,}500.$$

3. \$650,000. According to Problem 2, this money could be deposited into the bank. You could take out \$50,000 per year and still have \$72,500 left over after 20 years.

10.5. Modeling with Difference Equations

In this section we show how difference equations may be used to build mathematical models of a number of phenomena. Since the difference equation describing a situation contains as much data as the formula for y_n, we shall regard the difference equation itself as a mathematical model. We demonstrate in this section that the difference equation and a sketch of its graph can often be of more use than an explicit solution.

The concept of proportionality will be needed to develop some of these models.

Proportionality To say that two quantities are proportional is the same as saying that one quantity is equal to a constant times the other quantity. For instance, in a state having a 4% sales tax the sales tax on an item is proportional to the price of the item, since

$$[\text{sales tax}] = .04[\text{price}].$$

Here the constant of proportionality is .04. In general, for two proportional quantities we have

$$[\text{first quantity}] = k[\text{second quantity}],$$

where k is some fixed constant of proportionality. Note that if both quantities are positive and if the first quantity is known to always be smaller than the second, then k is a positive number less than 1; that is, $0 < k < 1$.

EXAMPLE 1 (*Radioactive decay*) Certain forms of natural elements, such as uranium 238, strontium 90, and carbon 14, are radioactive. That is, they decay, or dissipate, over a period of time. Physicists have found that this process of decay obeys the following law: Each year the amount which decays is proportional to the amount present at the start of the year. Construct a mathematical model describing radioactive decay.

Solution Let y_n = the amount left after n years. The physical law states that

$$[\text{amount that decays in year } n + 1] = k \cdot y_n,$$

where $k \cdot y_n$ represents a constant of proportionality times the amount present at the start of that year. Therefore,

$$\begin{array}{ccccc} y_{n+1} & = & y_n & - & ky_n \\ \begin{bmatrix}\text{amount after}\\ \text{year } n+1\end{bmatrix} & = & \begin{bmatrix}\text{amount after}\\ \text{year } n\end{bmatrix} & - & \begin{bmatrix}\text{amount that decays}\\ \text{in year } n+1\end{bmatrix}\end{array}$$

or

$$y_{n+1} = (1 - k)y_n,$$

where k is a constant between 0 and 1. This difference equation is the mathematical model for radioactive decay.

EXAMPLE 2 Experiment shows that for cobalt 60 (a radioactive form of cobalt used in cancer therapy) the constant k above is given by $k = .12$.

(a) Write the difference equation for y_n in this case.

(b) Sketch the graph of the difference equation.

Solution (a) Setting $k = .12$ in the result of the preceding example, we get

$$y_{n+1} = (1 - .12)y_n$$

or

$$y_{n+1} = .88y_n.$$

(b) Since .88 is positive and less than 1, the graph is monotonic and attracted to the line $y = b/(1 - a) = 0/(1 - .88) = 0$. The graph is sketched in Fig. 1.

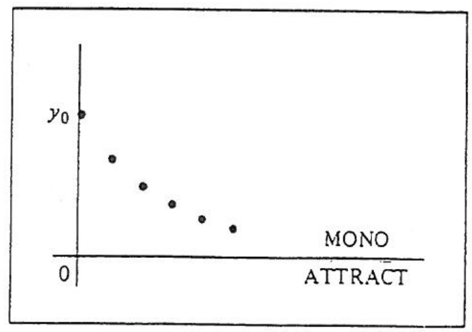


FIGURE 1

EXAMPLE 3 (*Growth of bacteria*) A bacteria culture grows in such a way that each hour the increase in the number of bacteria in the culture is proportional to the total number present at the beginning of the hour. Sketch a graph depicting the growth of the culture.

Solution Let y_n = the number of bacteria present after n hours. Then the increase during the next hour is ky_n, where k is a positive constant of proportionality. Therefore,

$$\underset{\begin{bmatrix}\text{number after}\\ n+1 \text{ hours}\end{bmatrix}}{y_{n+1}} = \underset{\begin{bmatrix}\text{number after}\\ n \text{ hours}\end{bmatrix}}{y_n} + \underset{\begin{bmatrix}\text{increase during}\\ (n+1)\text{st hour}\end{bmatrix}}{ky_n},$$

so

$$y_{n+1} = (1 + k)y_n.$$

This last equation gives a mathematical model of the growth of the culture. Since $a = 1 + k > 0$ and $|a| > 1$, the graph is monotonic and repelled from $b/(1 - a) = 0$. Thus, the graph is as drawn in Fig. 2.

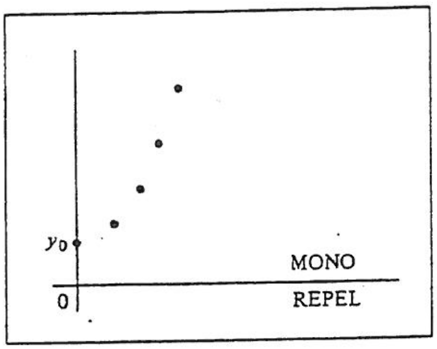


FIGURE 2

EXAMPLE 4 (*Spread of information*) Suppose that at 8 A.M. on a Saturday the local radio and TV stations in a town start broadcasting a certain piece of news. The number of people learning the news each hour is proportional to the number who had not yet heard it by the end of the preceding hour.

(a) Write down a difference equation describing the spread of the news through the population of the town.

(b) Sketch the graph of this difference equation for the case where the constant of proportionality is .3 and the population of the town is 50,000.

Solution The example states that two quantities are proportional. The first quantity is the number of people learning the news each hour and the second is the number who have not yet heard it by the end of the preceding hour.

(a) Let y_n = the number of people who have heard the news after n hours. The terms $y_0, y_1, y_2, \ldots$ are increasing.

Let P = the total population of the town. Then the number of people who have not yet heard the news after n hours is $P - y_n$. Thus, the assumption can be stated in the mathematical form

$$\underset{\begin{bmatrix}\text{number who know}\\ \text{after } n+1 \text{ hours}\end{bmatrix}}{y_{n+1}} = \underset{\begin{bmatrix}\text{number who know}\\ \text{after } n \text{ hours}\end{bmatrix}}{y_n} + \underset{\begin{bmatrix}\text{number who learn}\\ \text{during } (n+1)\text{st hour}\end{bmatrix}}{k(P - y_n)},$$

where k is a constant of proportionality. The constant k tells how fast the news is traveling. It measures the percentage of the uninformed population which hears the news each hour. It is clear that $0 < k < 1$.

For part (b) we assume that $k = .3$ and that the town has a population of 50,000. Further, measure the number of people in thousands. Then $P = 50$ and the mathematical model of the spread of the news is

$$\begin{aligned} y_{n+1} &= y_n + .3(50 - y_n) \\ &= y_n + 15 - .3y_n \\ &= .7y_n + 15. \end{aligned}$$

Suppose that initially no one had heard the news, so that

$$y_0 = 0.$$

Now we may sketch the graph:

$$\frac{b}{1-a} = \frac{15}{1-.7} = \frac{15}{.3} = 50.$$

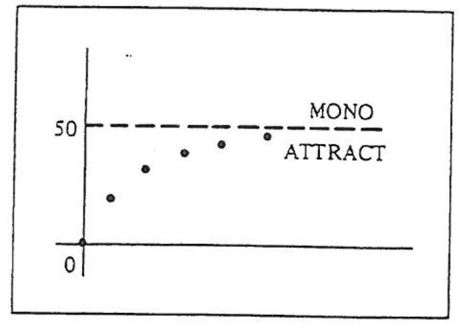


FIGURE 3

Since $a = .7$, the graph is monotonic and attracted to the line $y = 50$. Note that the number of people who hear the news increases and approaches the population of the entire town. The increases between consecutive terms are large at first, then become smaller. This coincides with the intuitive impression that the news spreads rapidly at first, then progressively slower.

EXAMPLE 5 (*Supply and demand*) This year's level of production and price for most agricultural products greatly affects the level of production and price next year. Suppose that the current crop of soybeans in a certain country is 80 million bushels. Let q_n denote the quantity* of soybeans grown n years from now, and let p_n denote the market price* in n years. Suppose experience has shown that q_n and p_n are related by the following equations:

$$\begin{aligned} p_n &= 20 - .1q_n \\ q_{n+1} &= 5p_n - 10. \end{aligned}$$

Draw a graph depicting the changes in production from year to year.

Solution What we seek is the graph of a difference equation for q_n.

$$\begin{aligned} q_{n+1} &= 5p_n - 10 \\ &= 5(20 - .1q_n) - 10 \\ &= 100 - .5q_n - 10 \\ &= -.5q_n + 90. \end{aligned}$$

* q_n in terms of millions of bushels, p_n in terms of dollars per bushel.

This is a difference equation for q_n with $a = -.5$, $b = 90$, $b/(1 - a) = 90/(1 - (-.5)) = 60$. Since a is negative, the graph is oscillating. Since $|a| = .5 < 1$, the graph is attracted to the line $y = b/(1 - a) = 60$. The initial condition is $q_0 = 80$. So the graph is as drawn in Fig. 4. Note that the graph oscillates and approaches 60. That is, the crop size fluctuates from year to year and eventually gets close to 60 million bushels.

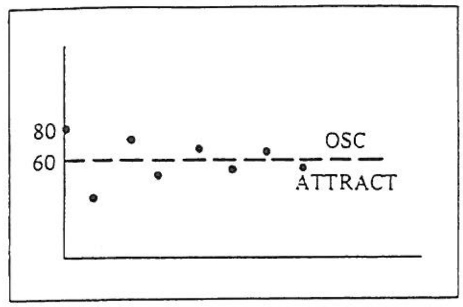


FIGURE 4

PRACTICE PROBLEMS 5

1. (*Glucose infusion*) Glucose is being given to a patient intravenously at the rate of 100 milligrams per minute. Let y_n be the amount of glucose in the blood after n minutes, measured in milligrams. Suppose that each minute the body takes from the blood 2% of the amount of glucose present at the beginning of that minute. Find a difference equation for y_n and sketch its graph. (Hint: Each minute the amount of glucose in the blood is increased by the intravenous infusion and decreased by the absorption into the body.)

2. (*Light at ocean depths*) Sunlight is absorbed by water and so, as one descends into the ocean, the intensity of light diminishes. Suppose that at each depth, going down one more meter causes a 20% decrease in the intensity of the sunlight. Find a difference equation for y_n, the intensity of the light at a depth of n meters, and sketch its graph.

EXERCISES 5

1. In a certain country with current population 100 million, each year the number of births is 3% and the number of deaths 1% of the population at the beginning of the year. Find the difference equation for y_n, the population after n years. Sketch its graph.

2. A small city with current population 50,000 is experiencing an emigration of 600 people each year. Assuming that each year the increase in population due to natural causes is 1% of the population at the start of that year, find the difference equation for y_n, the population after n years. Sketch its graph.

3. After a certain drug is injected, each hour the amount removed from the bloodstream by the body is 25% of the amount in the bloodstream at the beginning of the hour. Find the difference equation for y_n, the amount in the bloodstream after n hours, and sketch the graph.

4. The atmospheric pressure at sea level is 14.7 pounds per square inch. Suppose that at any elevation an increase of one mile results in a decrease of 20% of the

atmospheric pressure at that elevation. Find the difference equation for y_n, the atmospheric pressure at elevation n miles, and sketch its graph.

5. A sociological study* was made to examine the process by which doctors decide to adopt a new drug. Certain doctors who had little interaction with other physicians were called "isolated." Out of 100 isolated doctors, each month the number who adopted the new drug that month was 8% of those who had not yet adopted the drug at the beginning of the month. Find a difference equation for y_n, the number of isolated physicians using the drug after n months, and sketch its graph.

6. A cell is put into a fluid containing an 8 milligrams/liter concentration of a solute. (This concentration stays constant throughout.) Initially, the concentration of the solute in the cell is 3 milligrams/liter. The solute passes through the cell membrane at such a rate that each minute the increase in concentration in the cell is 40% of the difference between the outside concentration and the inside concentration. Find the difference equation for y_n, the concentration of the solute in the cell after n minutes, and sketch its graph.

7. Psychologists have found that in certain learning situations in which there is a maximum amount which can be learned, the additional amount learned each minute is proportional to the amount yet to be learned at the beginning of that minute. Let 12 units of information be the maximum amount that can be learned and let the constant of proportionality be 30%. Find a difference equation for y_n, the amount learned after n minutes, and sketch its graph.

8. Consider two genes A and a in a population, where A is a dominant gene and a is a recessive gene controlling the same genetic trait. (That is, A and a belong to the same locus.) Suppose initially 80% of the genes are A and 20% are a. Suppose that in each generation .003% of genes A mutate to gene a. Find a difference equation for y_n, the percentage of genes a after n generations, and sketch its graph. [Note: The percentage of genes A = 1 − (the percentage of genes a).]

9. Thirty thousand dollars is deposited in a savings account paying 5% interest compounded annually, and \$1000 is withdrawn from the account at the end of each year. Find the difference equation for y_n, the amount in the account after n years, and sketch its graph.

10. Rework Exercise 9 where \$15,000 is deposited initially.

11. When a cold object is placed in a warm room, each minute its increase in temperature is proportional to the difference between the room temperature and the temperature of the object at the beginning of the minute. Suppose that the room temperature is 70°, the initial temperature of the object is 40°, and the constant of proportionality is 20%. Find the difference equation for y_n, the temperature of the object after n minutes, and sketch its graph.

12. Suppose that the annual amount of electricity used in the United States will increase at a rate of 7% each year and that this year 2.6 trillion kilowatt-hours

* James S. Coleman, Elihu Katz, and Herbert Menzel: "The Diffusion of an Innovation among Physicians," *Sociometry*, 20 (1957), 253–270.

are being used. Find a difference equation for y_n, the number of kilowatt-hours to be used during the year that is n years from now, and sketch its graph.

13. Suppose that in Example 5 the current price of soybeans is \$10 per bushel. Find the difference equation for p_n and sketch its graph. [Note: Since $p_n = 20 - .1q_n$ holds for each year, it holds for the $(n + 1)$st year. That is, $p_{n+1} = 20 - .1q_{n+1}$.]

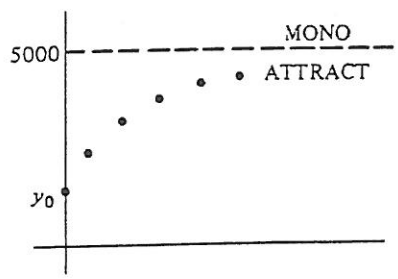
SOLUTIONS TO PRACTICE PROBLEMS 5

1. The amount of glucose in the blood is affected by two factors. It is being increased by the steady infusion of glucose; each minute the amount is increased by 100 milligrams. On the other hand, the amount is being decreased each minute by $.02y_n$. Therefore,

$$\begin{array}{ccccccc} y_{n+1} & = & y_n & + & 100 & - & .02y_n \\ \begin{bmatrix}\text{amount after}\\ n+1 \text{ minutes}\end{bmatrix} & = & \begin{bmatrix}\text{amount after}\\ n \text{ minutes}\end{bmatrix} & + & \begin{bmatrix}\text{increase due}\\ \text{to infusion}\end{bmatrix} & - & \begin{bmatrix}\text{amount taken}\\ \text{by the body}\end{bmatrix} \end{array}$$

or

$$y_{n+1} = .98y_n + 100.$$



In order to sketch the graph, first compute $b/(1 - a)$.

$$\frac{b}{1-a} = \frac{100}{1 - .98} = \frac{100}{.02} = 5000.$$

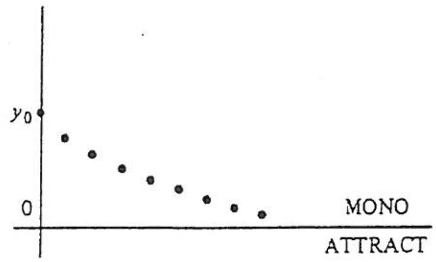
The amount of glucose in the blood rises and approaches 5000 milligrams.

2. The change in intensity in going from depth n meters to $n + 1$ meters is 20% of the intensity at n meters. Therefore,

$$\begin{array}{ccccc} y_{n+1} & = & y_n & - & .20y_n \\ \begin{bmatrix}\text{intensity at}\\ n+1 \text{ meters}\end{bmatrix} & = & \begin{bmatrix}\text{intensity at}\\ n \text{ meters}\end{bmatrix} & - & \begin{bmatrix}\text{change in}\\ \text{intensity}\end{bmatrix} \end{array}$$

$$y_{n+1} = .8y_n.$$

Thus $b/(1 - a) = 0$, and the graph is as follows.



Chapter 10: CHECKLIST

- □ Difference equation
- □ Initial value
- □ Graph of difference equation
- □ Solution of difference equation for $a \neq 1$
- □ Solution of difference equation for $a = 1$
- □ Simple interest
- □ Compound interest
- □ Increasing and decreasing graphs
- □ Monotonic and oscillating graphs
- □ Constant graph
- □ Technique for sketching graph
- □ Mortgage
- □ Annuity

Chapter 10: SUPPLEMENTARY EXERCISES

1. Consider the difference equation $y_{n+1} = -3y_n + 8$, $y_0 = 1$.

   (a) Generate y_1, y_2, y_3 from the difference equation.

   (b) Solve the difference equation.

   (c) Use the solution in (b) to obtain y_4.

2. Consider the difference equation $y_{n+1} = y_n - \frac{3}{2}$, $y_0 = 10$.

   (a) Generate y_1, y_2, y_3 from the difference equation.

   (b) Solve the difference equation.

   (c) Use the solution in (b) to obtain y_6.

3. How much money would you have to put into a savings account initially at 8% interest compounded quarterly in order to have $6600 after 10 years? [Note: $(1.08)^{10} \approx 2.16$; $(1.08)^{40} \approx 21.7$; $(1.02)^{10} \approx 1.2$; $(1.02)^{40} \approx 2.2$.]

4. How much money would you have in the bank after two years if you deposited \$1000 at 5.2% interest compounded weekly? [Note: $(1.052)^2 \approx 1.107$; $(1.001)^{104} \approx 1.110$; $(1.12)^2 \approx 1.254$; $(1.02)^{52} \approx 2.800$.]

5. Make a rough sketch of the graph of the difference equation $y_{n+1} = -\frac{1}{3}y_n + 8$, $y_0 = 10$ without generating terms.

6. Make a rough sketch of the graph of the difference equation $y_{n+1} = 1.5y_n - 2$, $y_0 = 5$ without generating terms.

7. The population of a certain city is currently 120,000. The growth rate of the population due to an excess of births to deaths is 3% per year. Furthermore, each year 600 people are moving out of the city. Let y_n be the population after n years.

   (a) Find the difference equation for the population growth.

   (b) What will be the population after 20 years? [Note: $(1.03)^{20} \approx 1.8$.]

8. The monthly payment on a \$35,000 30-year mortgage at 12% interest compounded monthly is \$360. Let y_n be the unpaid balance of the mortgage after n months.

   (a) Give the difference equation expressing y_{n+1} in terms of y_n. Also, give y_0.

   (b) What is the unpaid balance after seven years? [Note: $(1.12)^7 \approx 2.21$; $(1.12)^{84} \approx 13{,}624$; $(1.01)^7 \approx 1.07$; $(1.01)^{84} \approx 2.3$.]

9. How much money must be deposited at the end of each week into an annuity at $5\frac{1}{5}$% (= .052) interest compounded weekly in order to have \$40,000 after 21 years? [Note: $(1.001)^{21} \approx 1.02$; $(1.001)^{1092} \approx 3$.]

10. Find the monthly payment on a \$33,100 20-year mortgage at 6% interest compounded monthly. [Note: $(1.005)^{240} \approx 3.31$.]

11. How much money can you borrow at 8% interest compounded annually if the loan is to be paid off in yearly installments for 18 years and you can afford to pay \$2400 per year? [Note: $(1.08)^{18} \approx 4$.]

12. A college alumnus pledges to give his alma mater \$5 at the end of each month for four years. If the college puts the money in a savings account earning 6% interest compounded monthly, how much money will be in the account at the end of four years? [Note: $(1.06)^{48} \approx 16.39$; $(1.06)^4 \approx 1.26$; $(1.005)^{48} \approx 1.27$; $(1.005)^4 \approx 1.02$.]

13. An unknown candidate for governor in a state having 1,000,000 voters mounts an extensive media campaign. Each day, 10% of the voters who do not yet know about him become aware of his candidacy. Let y_n be the number of voters who are aware of his candidacy after n days. Find a difference equation for y_n and sketch its graph.

14. Suppose that 100 people were just elected to a certain state legislature and that after each term 8% of those still remaining from this original group will either retire or not be reelected. Let y_n be the number of legislators from the original group of 100 who are still serving after n terms. Find a difference equation for y_n and sketch its graph.

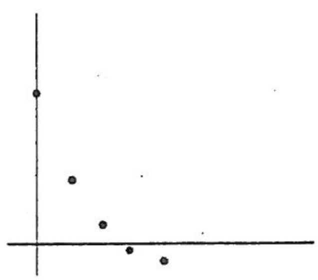
Answers to Odd-numbered Problems

CHAPTER 10
EXERCISES 1, *page* 383

1. $4, -6; 2$ 3. $-\frac{1}{2}, 0; 0$ 5. $-\frac{2}{3}, 15; 9$

7. (a) $10, 4, 1, -\frac{1}{2}, -\frac{5}{4}$

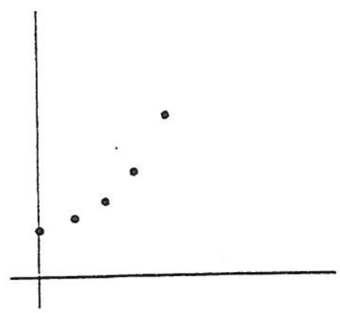
(b)



(c) $y_n = -2 + 12(\frac{1}{2})^n$

9. (a) $3.5, 4, 5, 7, 11$

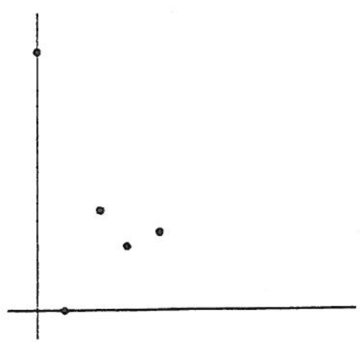
(b)



(c) $y_n = 3 + (.5)2^n$

11. (a) $17.5, 0, 7, 4.2, 5.32$

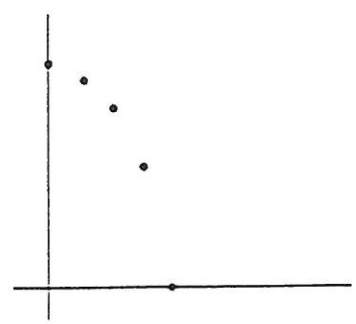
(b)



(c) $y_n = 5 + (12.5)(-.4)^n$

13. (a) $15, 14, 12, 8, 0$

(b)

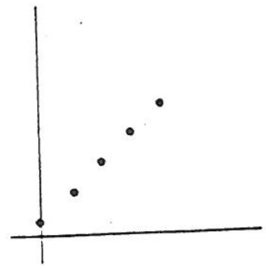


(c) $y_n = 16 - 2^n$

15. $1, 5, 5.8, 5.96, 5.992$ 17. $y_{n+1} = 1.05y_n, y_0 = 1000$

19. $y_{n+1} = 1.05y_n + 100, y_0 = 1000$

21. (a) 1, 3, 5, 7, 9 (b)



(c) Since $a = 1$, $1 - a = 0$ and so $\frac{b}{1-a} = \frac{2}{0}$ which is not defined

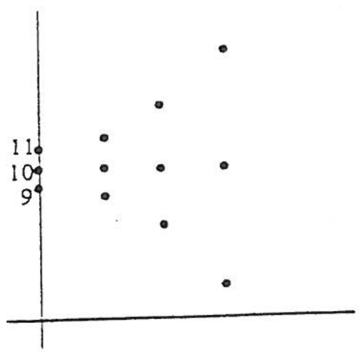
23. $30.

EXERCISES 2, *page* 392

1. $y_n = 1 + 5n$ 3. $80(1.0075)^{60}$ 5. $80(1 + \frac{1}{365})^{1825}$

7. 108 9. $A\left(1 + \frac{r}{k}\right)^{kt}$

11.



Notice that $\frac{b}{1-a} = \frac{-10}{1-2} = 10$ and that when $y_0 \neq 10$, the terms are repelled (that is, move away) from the line $y = 10$.

13. $y_n = 5 + 2(.4)^n$; y_n approaches 5

15. $y_n = 2(-5)^n$; y_n gets arbitrarily large and alternates between being positive and negative

17. $y_{n+1} = 1.0075y_n - 350$, $y_0 = 38{,}900$

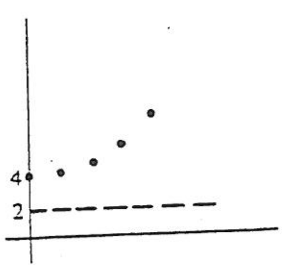
19. $y_{n+1} = y_n - 2000$, $y_0 = 50{,}000$; $y_n = 50{,}000 - 2000n$

EXERCISES 3, *page* 402

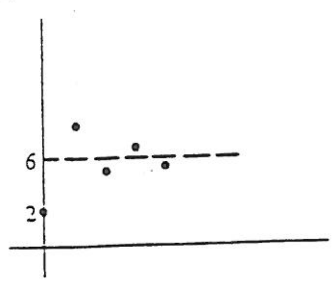
1. A, B, D, F, H 3. B, D, E, F

5. B, D, E, F 7. A, C, H, (possibly G)

9.



11.



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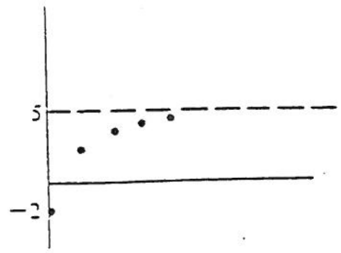

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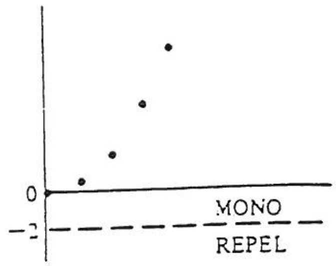
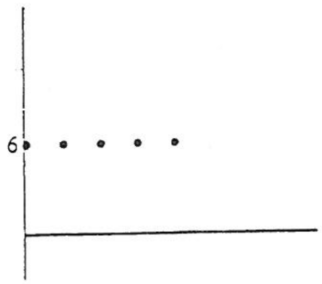


17.



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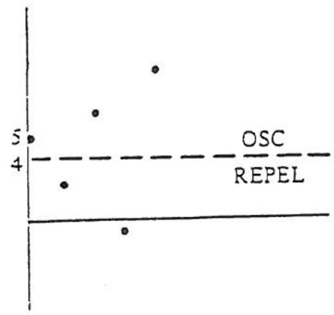

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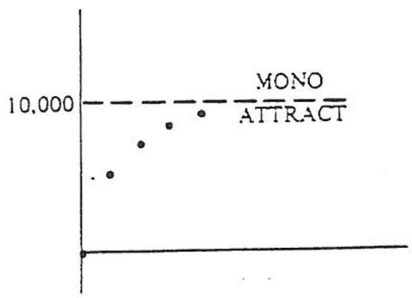

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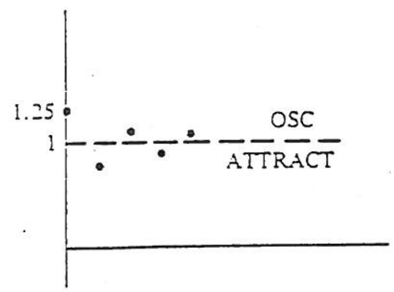


25. Less than \$60,000

27. (a) $y_{n+1} = 1.06y_n - 120$ (b) At least \$2000

EXERCISES 4, *page* 409

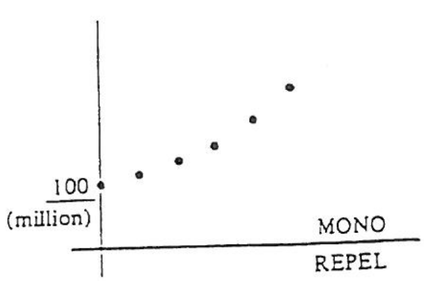
1. $y_{n+1} = 1.0075y_n - 261.50,\ y_0 = 32{,}500$

3. $y_{n+1} = 1.015y_n + 200,\ y_0 = 4000$ 5. \$46,000 7. \$11,000

9. \$2000 11. \$133.02

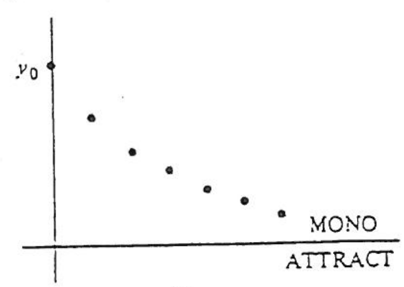
EXERCISES 5, *page* 414

1.



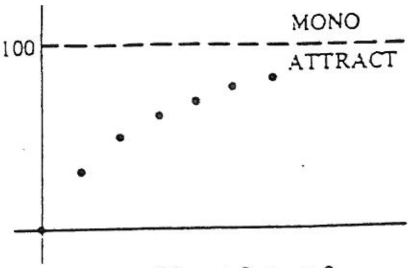
$y_{n+1} = 1.02y_n,\ y_0 = 100$

3.



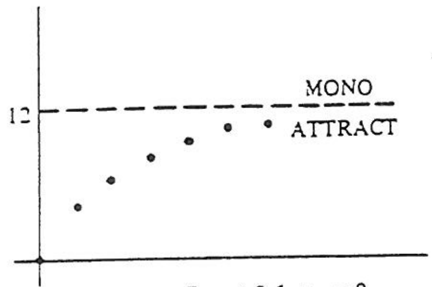
$y_{n+1} = .75y_n$

5.



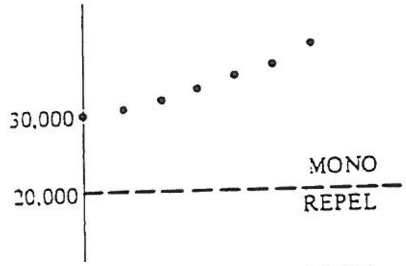
$y_{n+1} = .92y_n + 8,\ y_0 = 0$

7.



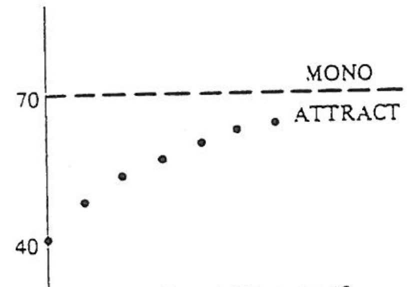
$y_{n+1} = .7y_n + 3.6,\ y_0 = 0$

9.



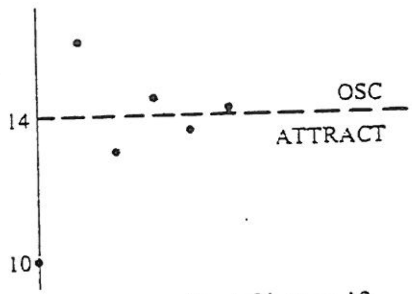
$y_{n+1} = 1.05y_n - 1000,\ y_0 = 30{,}000$

11.



$y_{n+1} = .8y_n + 14,\ y_0 = 40$

13.



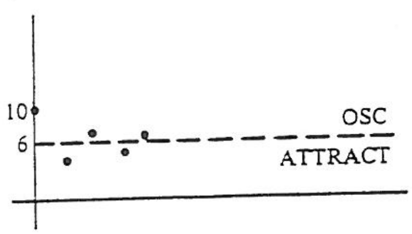
$p_{n+1} = -.5p_n + 21,\ p_0 = 10$

CHAPTER 10: SUPPLEMENTARY EXERCISES, *page* 417

1. (a) 5, −7, 29 (b) $y_n = 2 - (-3)^n$ (c) −79

3. $3,000

5.

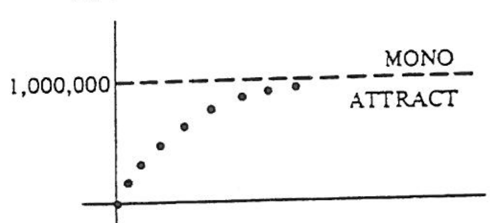


7. (a) $y_{n+1} = 1.03y_n - 600,\ y_0 = 120{,}000$ (b) 200,000

9. $20

11. $22,500

13.



$y_{n+1} = .9y_n + 100{,}000$